GEOCENTRIC EQUATORIAL EPHEMERIS OF A COMET, REFERRED TO
THE MEAN EQUATOR AND EQUINOX OF J2000.0

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This document demonstrates the programming power of Mathcad PLUS 6 by developing a live procedure which calculates a geocentric equatorial ephemeris for a comet using Comet Hale-Bopp as an example. [An *ephemeris* is a table of times and sky coordinates, e.g., right ascension (R.A.) and declination (Dec.) coordinates of a celestial body, at those times.]

By changing the orbital elements and other input data in Step 12, below, you can generate a geocentric equatorial ephemeris for any comet, for any time period of interest. When you have generated an ephemeris for the comet, you can input the sky coordinates to Martin V. Zombeck's Mathcad electronic book, *Astronomical Formulas* (see Chapter 3, Astronomical Phenomena), along with your latitude and longitude, to calculate rise and set times for the comet. Or, you can simply identify on a planisphere ("star wheel") the constellation in which the comet lies, and use the planisphere to determine approximate local times of rise and set.

1. Define a vector function, **C**, to calculate the first four c-functions.

$C(x) :=$

$N \leftarrow 0$

while $|x| \geq 0.1$

  $x \leftarrow \frac{x}{4}$

  $N \leftarrow N+1$

$c_3 \leftarrow \frac{1}{6}\cdot\left(1-\frac{x}{20}\cdot\left(1-\frac{x}{42}\cdot\left(1-\frac{x}{72}\cdot\left(1-\frac{x}{110}\cdot\left(1-\frac{x}{156}\cdot\left(1-\frac{x}{210}\right)\right)\right)\right)\right)\right)$

$c_2 \leftarrow \frac{1}{2}\cdot\left(1-\frac{x}{12}\cdot\left(1-\frac{x}{30}\cdot\left(1-\frac{x}{56}\cdot\left(1-\frac{x}{090}\cdot\left(1-\frac{x}{132}\cdot\left(1-\frac{x}{182}\right)\right)\right)\right)\right)\right)$

$c_1 \leftarrow 1 - c_3 \cdot x$

$c_0 \leftarrow 1 - c_2 \cdot x$

while $N > 0$

  $N \leftarrow N-1$

  $c_3 \leftarrow \frac{\left(c_1 \cdot c_2 + c_3\right)}{4}$

  $c_2 \leftarrow \frac{c_1 \cdot c_1}{2}$

  $c_1 \leftarrow c_1 \cdot c_0$

  $c_0 \leftarrow 2 \cdot c_0 \cdot c_0 - 1$

c

2. Define uniform, two-body path propagator function, **U2PM**.

$k := 0.01720209895$ $\mu := 1.0$

$K := k \cdot \sqrt{\mu}$

$$U2PM(K, q, e, i, \Omega, \omega, \Delta t) :=$$

$$\alpha \leftarrow K^2 \cdot \frac{(1-e)}{q}$$
$$p \leftarrow q \cdot (1+e)$$
$$s \leftarrow \frac{\Delta t}{q}$$
$$\Delta s \leftarrow s$$
while $|\Delta s| \geq 0.00000001$

$$c \leftarrow C(\alpha \cdot s^2)$$
$$f \leftarrow q \cdot s + K^2 \cdot e \cdot s^3 \cdot c_3 - \Delta t$$
$$Df \leftarrow q + K^2 \cdot e \cdot s^2 \cdot c_2$$
$$DDf \leftarrow K^2 \cdot e \cdot s \cdot c_1$$
if $Df \geq 0$

$m \leftarrow 1$

else

$m \leftarrow -1$

$$\Delta s \leftarrow \frac{-5 \cdot f}{\left(Df + m \cdot \sqrt{\left|(4 \cdot Df)^2 - 20 \cdot f \cdot DDf\right|}\right)}$$
$$s \leftarrow s + \Delta s$$

$$P_0 \leftarrow \cos(\Omega) \cdot \cos(\omega) - \sin(\Omega) \cdot \cos(i) \cdot \sin(\omega)$$
$$P_1 \leftarrow \sin(\Omega) \cdot \cos(\omega) + \cos(\Omega) \cdot \cos(i) \cdot \sin(\omega)$$
$$P_2 \leftarrow \sin(i) \cdot \sin(\omega)$$
$$Q_0 \leftarrow -(\cos(\Omega) \cdot \sin(\omega) + \sin(\Omega) \cdot \cos(i) \cdot \cos(\omega))$$
$$Q_1 \leftarrow -(\sin(\Omega) \cdot \sin(\omega) - \cos(\Omega) \cdot \cos(i) \cdot \cos(\omega))$$
$$Q_2 \leftarrow \sin(i) \cdot \cos(\omega)$$
$$rcosv \leftarrow q - K^2 \cdot s^2 \cdot c_2$$
$$rsinv \leftarrow K \cdot \sqrt{p} \cdot s \cdot c_1$$
$$rcosv \cdot P + rsinv \cdot Q$$

3. Test the functions **U2PM** and **C** with data from orbit of Comet Hale-Bopp, near perihelion of 1997 March 31.95962 TT (1997 day 90.95962), using 1997 January 1.0 TT as date of interest (1997 day 1.0).

$q := 0.9143839$ $\quad e := 0.9952982$ $\quad DegPerRad := \dfrac{180}{\pi}$

$i := \dfrac{089.43088}{DegPerRad}$ $\quad \Omega := \dfrac{282.47058}{DegPerRad}$ $\quad \omega := \dfrac{130.56797}{DegPerRad}$

$T := 90.95962$ $\quad \Delta t := 1.0 - T$

$r_C := U2PM(K, q, e, i, \Omega, \omega, \Delta t)$ $\quad r_C = \begin{bmatrix} 0.2881055936 \\ -1.2478104851 \\ 1.1937843701 \end{bmatrix}$

4. Calculate the heliocentric ecliptic coordinates of the Earth-Moon barycenter, referred to the mean ecliptic and equinox of J2000 (Julian Date 2451545.0 TT), using data from the *Explanatory Supplement to the Astronomical Almanac* (P. Kenneth Seidelmann, Editor, University Science Books, 1992), p. 316.

$JD := 2450449.5$ $\quad JD_o := 2451545.0$

$\Delta T := \dfrac{JD - JD_o}{36525.0}$ $\quad a := 1.00000011 - 0.00000005 \cdot \Delta T$

$e := 0.01671022 - 0.00003804 \cdot \Delta T$

$q := a \cdot (1 - e)$

$\mu := 1.00000304$ $\quad K := k \cdot \sqrt{\mu}$ $\quad n := K \cdot a^{\frac{-3}{2}}$

$SecPerDeg := 3600.0$

$SecPerRev := SecPerDeg \cdot 360.0$

$$i := \frac{0.00005 - \frac{46.94 \cdot \Delta T}{SecPerDeg}}{DegPerRad}$$

$$\Omega := 0.0$$

$$\omega := \frac{102.94719 + \frac{1198.28 \cdot \Delta T}{SecPerDeg}}{DegPerRad}$$

$$L := \frac{100.46435 + \frac{1293740.63 + 99 \cdot SecPerRev}{SecPerDeg} \cdot \Delta T}{DegPerRad}$$

$$T := JD + \frac{-\mathrm{mod}\left(L - \omega, 2 \cdot \pi\right)}{n}$$

$$T = 2450451.73986008$$

$$\Delta t := JD - T$$

$$\Delta t = -2.2398600751$$

$$r_{EM} := U2PM\left(K, q, e, i, \Omega, \omega, \Delta t\right)$$

$$r_{EM} = \begin{bmatrix} -0.1817945886 \\ 0.966350205 \\ 0.0000074392 \end{bmatrix}$$

5. Correct the heliocentric position of the Earth-Moon barycenter to the geocenter.

$$L_M := \frac{mod\left(218.0 + 481268.0 \cdot \Delta T, 360.0\right)}{DegPerRad}$$

$$r_{EM} := \begin{bmatrix} r_{EM_0} - 0.0000312 \cdot cos\left(L_M\right) \\ r_{EM_1} - 0.0000312 \cdot sin\left(L_M\right) \\ r_{EM_2} \end{bmatrix}$$

6. Calculate the geocentric ecliptic coordinates of the comet, then transform from geocentric ecliptic to geocentric equatorial coordinates.

$$r := r_C - r_{EM}$$

$$\varepsilon := \frac{23.4392911}{DegPerRad}$$

$$M := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\varepsilon) & -\sin(\varepsilon) \\ 0 & \sin(\varepsilon) & \cos(\varepsilon) \end{bmatrix} \qquad r := M \cdot r$$

$$\alpha := angle\left(r_0, r_1\right) \cdot \frac{DegPerRad}{15} \qquad \alpha = 18.7078822326$$

$$\delta := asin\left(\frac{r_2}{\sqrt{\left(r_0\right)^2 + \left(r_1\right)^2 + \left(r_2\right)^2}}\right) \cdot DegPerRad \qquad \delta = 4.8088857748$$

The foregoing live equations demonstrate the effectiveness of the functions **U2PM** and **C**, both for propagating a highly eccentric orbit, the heliocentric orbit of Comet Hale-Bopp, and for propagating a low-eccentricity orbit, the heliocentric orbit of the Earth-Moon barycenter. Function **U2PM** will also, without modification, propagate a parabolic or a hyperbolic trajectory, and that is the key to its power as a "universal variables" method.

It should be further noted that the foregoing calculations constitute a complete "geocentric equatorial ephemeris point" calculation for a single position of Comet Hale-Bopp on the celestial sphere.

Since the goal of this worksheet is to produce a concise and economical specification of how to generate a geocentric equatorial ephemeris for a comet, taking full advantage of the programming power of Mathcad, we will now define two more functions in terms of the equations just developed.

7. Define a function, **HGEO**, to calculate the heliocentric ecliptic position of the geocenter as a function of the Julian date, with epoch at 2000 January 1.5 TT (JD = 2451545.0). Note that k (as defined in Step 2) and DegPerRad (as defined in Step 3) are "global" arguments of this function, i.e., they are defined in the worksheet outside of the function, and prior to its definition. So also are SecPerDeg and SecPerRev (as defined in Step 4).

$$HGEO(JD) := \left\| \begin{array}{l} JD_o \leftarrow 2451545.0 \\ T_c \leftarrow \dfrac{JD - JD_o}{36525.0} \\ a \leftarrow 1.00000011 - 0.00000005 \cdot T_c \\ e \leftarrow 0.01671022 - 0.00003804 \cdot T_c \\ q \leftarrow a \cdot (1 - e) \\ \mu \leftarrow 1.00000304 \\ K \leftarrow k \cdot \sqrt{\mu} \\ n \leftarrow K \cdot a^{\frac{-3}{2}} \\ \omega \leftarrow \dfrac{102.94719 + \dfrac{1198.28 \cdot T_c}{SecPerDeg}}{DegPerRad} \\ i \leftarrow \dfrac{0.00005 - \dfrac{46.94 \cdot T_c}{SecPerDeg}}{DegPerRad} \\ \Omega \leftarrow 0.0 \\ L \leftarrow \dfrac{100.46435 + \dfrac{1293740.63 + 99 \cdot SecPerRev}{SecPerDeg} \cdot T_c}{DegPerRad} \\ T \leftarrow JD - \dfrac{\bmod(L - \omega, 2 \cdot \pi)}{n} \\ \Delta t \leftarrow JD - T \\ r_{EM} \leftarrow U2PM(K, q, e, i, \Omega, \omega, \Delta t) \\ L_M \leftarrow \dfrac{\bmod(218.0 + 481268.0 \cdot T_c, 360.0)}{DegPerRad} \\ \begin{bmatrix} r_{EM_0} - 0.0000312 \cdot \cos(L_M) \\ r_{EM_1} - 0.0000312 \cdot \sin(L_M) \\ r_{EM_2} \end{bmatrix} \end{array} \right.$$

8. Define function **ECEQ** to convert from geocentric ecliptic coordinates to geocentric equatorial coordinates at the J2000 epoch.

$$ECEQ(r) := \left\| \begin{array}{l} \varepsilon \leftarrow \dfrac{23.4392911}{DegPerRad} \\ M \leftarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\varepsilon) & -\sin(\varepsilon) \\ 0 & \sin(\varepsilon) & \cos(\varepsilon) \end{bmatrix} \\ M \cdot r \end{array} \right.$$

9. Now we can specify a comet ephemeris generation function, **ECOM**.

$$ECOM(q, e, i, \Omega, \omega, T, JD, \Delta d, N) := \left\| \begin{array}{l} \text{for } j \in 0..N \\ \quad \left\| \begin{array}{l} \Delta t \leftarrow JD - T \\ K \leftarrow k \\ r \leftarrow U2PM(K, q, e, i, \Omega, \omega, \Delta t) \\ r \leftarrow r - HGEO(JD) \\ r \leftarrow ECEQ(r) \\ \alpha \leftarrow \text{angle}\left(r_0, r_1\right) \cdot \dfrac{DegPerRad}{15} \\ \delta \leftarrow \text{asin}\left(\dfrac{r_2}{\sqrt{\left(r_0\right)^2 + \left(r_1\right)^2 + \left(r_2\right)^2}}\right) \cdot DegPerRad \\ \text{if } j = 0 \\ \quad \| \; Table \leftarrow [JD \;\; \alpha \;\; \delta] \\ \text{else} \\ \quad \| \; Table \leftarrow \text{stack}(Table, [JD \;\; \alpha \;\; \delta]) \\ JD \leftarrow JD + \Delta d \end{array} \right. \\ Table \end{array} \right.$$

10. Use function **ECOM** to generate an ephemeris for Comet Hale-Bopp, starting at 1997 March 17 TT (JD = 2450524.5), with 12 additional ephemeris points spaced 5 days apart.

$q := 0.9143839$ 1. Perihelion distance, A.U.

$e := 0.9952982$ 2. Orbital eccentricity.

$i := \dfrac{089.43088}{DegPerRad}$ 3. Orbital inclination, degrees.

$\Omega := \dfrac{282.47058}{DegPerRad}$ 4. Celestial longitude of ascending node, degrees.

$\omega := \dfrac{130.56797}{DegPerRad}$ 5. Argument of perihelion, degrees.

$JD_o := 2450448.5$ 6. Julian date for 1997 Jan 0.0 TT.

$T := JD_o + 90.95962$ 7. Julian date of perihelion passage.

$JD := JD_o + 76.0$ 8. Julian date of ephemeris start.

$\Delta d := 5.0$ 9. Ephemeris step size, in days.

$N := 12$ 10. Number of time steps to take.

$ECOM(q, e, i, \Omega, \omega, T, JD, \Delta d, N) =$

JD	Rt. Ascension	Declination
2450524.5	23.3536802299	43.9960020502
2450529.5	0.1668493701	45.5536315945
2450534.5	1.0068310373	45.7095811622
2450539.5	1.8071609033	44.4886930692
2450544.5	2.5184364833	42.1799437843
2450549.5	3.1223120948	39.1800289708
2450554.5	3.6243291357	35.8457740569
2450559.5	4.0406541822	32.4315064943
2450564.5	4.3891007163	29.0913758099
2450569.5	4.6852439709	25.9052985263
2450574.5	4.9414241319	22.9051982871
2450579.5	5.1669635525	20.094750595
		⋮

This ephemeris can be used with Martin Zombeck's Mathcad electronic book, *Astronomical Formulas* [11], or with a planisphere, to generate rise and set times on a date of interest.

However, we are not done, as there is an important issue yet to deal with: how good are the predictions, assuming that the orbital elements are good?

To determine how good the predictions are, we can compare them with output obtained from Montenbruck and Pfleger's COMET program [5]. But before we can do that, we will need to format the output right ascensions into hours, minutes, seconds, and tenths of seconds of time, and format the output declinations into degrees, minutes, and seconds or arc.

To do this, we will want to define a function to format the ephemeris. We let

$$M := ECOM(q, e, i, \Omega, \omega, T, JD, \Delta d, N)$$

and then operate on the matrix **M** with a formatting function, as defined in the next step.

11. Define a function, **FORM**, that formats the right ascensions and declinations produced by the comet ephemeris generation function, **ECOM**. Note that **M** is the input ephemeris matrix and N is the number of ephemeris points (rows) of **M**.

$$
\begin{aligned}
FORM(M,N) :=\ & \text{for } j \in 0..N \\
& \quad h_r \leftarrow M_{j,1} + \frac{0.5}{36000} \\
& \quad h \leftarrow \text{floor}(h_r) \\
& \quad m \leftarrow 60 \cdot (h_r - h) \\
& \quad s \leftarrow \frac{\text{floor}(600 \cdot (m - \text{floor}(m)))}{10} \\
& \quad m \leftarrow \text{floor}(m) \\
& \quad H_{j,0} \leftarrow h \\
& \quad P_{j,0} \leftarrow m \\
& \quad S_{j,0} \leftarrow s \\
& A \leftarrow \text{augment}(H, P) \\
& A \leftarrow \text{augment}(A, S) \\
& \text{for } j \in 0..N \\
& \quad d_r \leftarrow \left|M_{j,2}\right| + \frac{0.5}{3600} \\
& \quad d \leftarrow \text{floor}(d_r) \\
& \quad m \leftarrow 60 \cdot (d_r - d) \\
& \quad s \leftarrow \text{floor}(60 \cdot (m - \text{floor}(m))) \\
& \quad m \leftarrow \text{floor}(m) \\
& \quad H_{j,0} \leftarrow d \\
& \quad \text{if } M_{j,2} < 0 \\
& \quad\quad H_{j,0} \leftarrow -d \\
& \quad P_{j,0} \leftarrow m \\
& \quad S_{j,0} \leftarrow s \\
& A \leftarrow \text{augment}(A, H) \\
& A \leftarrow \text{augment}(A, P) \\
& A \leftarrow \text{augment}(A, S) \\
& A \leftarrow \text{augment}(M^{\langle 0 \rangle}, A)
\end{aligned}
$$

Application of the function **FORM** to the Nx7 matrix **M** gives the following results:

Julian Date	R.A. hr	R.A. mm	R.A. ss.s	Dec. dg	Dec. mm	Dec. ss
2450524.5	23	21	13.2	43	59	46
2450529.5	0	10	0.7	45	33	13
2450534.5	1	0	24.6	45	42	34
2450539.5	1	48	25.8	44	29	19
2450544.5	2	31	6.4	42	10	48
2450549.5	3	7	20.3	39	10	48
2450554.5	3	37	27.6	35	50	45
2450559.5	4	2	26.4	32	25	53
2450564.5	4	23	20.8	29	5	29
2450569.5	4	41	6.9	25	54	19
2450574.5	4	56	29.1	22	54	19
2450579.5	5	10	1.1	20	5	41
2450584.5	5	22	7.6	17	27	46

$$FORM(M,N)=\begin{bmatrix} 2450524.5 & 23 & 21 & 13.2 & 43 & 59 & 46 \\ 2450529.5 & 0 & 10 & 0.7 & 45 & 33 & 13 \\ 2450534.5 & 1 & 0 & 24.6 & 45 & 42 & 34 \\ 2450539.5 & 1 & 48 & 25.8 & 44 & 29 & 19 \\ 2450544.5 & 2 & 31 & 6.4 & 42 & 10 & 48 \\ 2450549.5 & 3 & 7 & 20.3 & 39 & 10 & 48 \\ 2450554.5 & 3 & 37 & 27.6 & 35 & 50 & 45 \\ 2450559.5 & 4 & 2 & 26.4 & 32 & 25 & 53 \\ 2450564.5 & 4 & 23 & 20.8 & 29 & 5 & 29 \\ 2450569.5 & 4 & 41 & 6.9 & 25 & 54 & 19 \\ 2450574.5 & 4 & 56 & 29.1 & 22 & 54 & 19 \\ 2450579.5 & 5 & 10 & 1.1 & 20 & 5 & 41 \\ 2450584.5 & 5 & 22 & 7.6 & 17 & 27 & 46 \end{bmatrix}$$

Montenbruck and Pfleger's COMET program gives these results for the same input data:

Julian Date	R.A. hr	R.A. mm	R.A. ss.s	Dec. dg	Dec. mm	Dec. ss
2450524.5	23	21	11.4	43	59	42
2450529.5	0	09	58.5	45	33	14
2450534.5	1	0	22.4	45	42	41
2450539.5	1	48	23.6	44	29	31
2450544.5	2	31	4.4	42	11	4
2450549.5	3	7	18.7	39	11	7
2450554.5	3	37	26.2	35	51	6
2450559.5	4	2	25.2	32	26	17
2450564.5	4	23	19.8	29	5	53
2450569.5	4	41	6.1	25	54	43
2450574.5	4	56	28.4	22	54	43
2450579.5	5	10	0.5	20	6	5
2450584.5	5	22	7.1	17	28	9

$$MPF:=\begin{bmatrix} 2450524.5 & 23 & 21 & 11.4 & 43 & 59 & 42 \\ 2450529.5 & 0 & 09 & 58.5 & 45 & 33 & 14 \\ 2450534.5 & 1 & 0 & 22.4 & 45 & 42 & 41 \\ 2450539.5 & 1 & 48 & 23.6 & 44 & 29 & 31 \\ 2450544.5 & 2 & 31 & 4.4 & 42 & 11 & 4 \\ 2450549.5 & 3 & 7 & 18.7 & 39 & 11 & 7 \\ 2450554.5 & 3 & 37 & 26.2 & 35 & 51 & 6 \\ 2450559.5 & 4 & 2 & 25.2 & 32 & 26 & 17 \\ 2450564.5 & 4 & 23 & 19.8 & 29 & 5 & 53 \\ 2450569.5 & 4 & 41 & 6.1 & 25 & 54 & 43 \\ 2450574.5 & 4 & 56 & 28.4 & 22 & 54 & 43 \\ 2450579.5 & 5 & 10 & 0.5 & 20 & 6 & 5 \\ 2450584.5 & 5 & 22 & 7.1 & 17 & 28 & 9 \end{bmatrix}$$

Comparison of our formatted output with that from Montenbruck & Pfleger's program shows larger differences than we would like (up to 2.2 seconds of time in R.A. and up to 24 seconds of arc in Dec.). There are two sources of differences:

a. Montenbruck & Pfleger's COMET program corrects the heliocentric motion of the comet for light-time, i.e., for the motion of the comet in the time it takes light from the comet to reach Earth.

b. Montenbruck & Pfleger account for planetary perturbations of the Earth-Moon system in their model for the orbital motion of the geocenter around the sun (see SUN200, pp. 23-26).

The second source of differences we can do nothing about, for we have adopted the mean elements model in the *Explanatory Supplement to the Astronomical Almanac* .

To eliminate the first source of differences, we will correct for light-time as per Montenbruck & Pfleger, pp. 76-77. To do this, we will need to define **U2PV**, an improved version of **U2PM**, which will calculate velocity as well as position.

First we find it convenient to define a preliminary function, **PQEQ**, which performs the necessary Euler angle transformations. This will keep **U2PV** from becoming too long to fit into a single printed page of Mathcad output.

$$PQEQ(i,\Omega,\omega,p,q) := \left\| \begin{array}{l} P_0 \leftarrow \cos(\Omega)\cdot\cos(\omega) - \sin(\Omega)\cdot\cos(i)\cdot\sin(\omega) \\ P_1 \leftarrow \sin(\Omega)\cdot\cos(\omega) + \cos(\Omega)\cdot\cos(i)\cdot\sin(\omega) \\ P_2 \leftarrow \sin(i)\cdot\sin(\omega) \\ Q_0 \leftarrow -(\cos(\Omega)\cdot\sin(\omega) + \sin(\Omega)\cdot\cos(i)\cdot\cos(\omega)) \\ Q_1 \leftarrow -(\sin(\Omega)\cdot\sin(\omega) - \cos(\Omega)\cdot\cos(i)\cdot\cos(\omega)) \\ Q_2 \leftarrow \sin(i)\cdot\cos(\omega) \\ p\cdot P + q\cdot Q \end{array} \right.$$

Then we define a function that performs the light-time correction, for incorporation into a new comet ephemeris generation function, **FCOM**.

$$LTIM(PV) := \left\| \begin{array}{l} r \leftarrow PV^{\langle 0 \rangle} \\ v \leftarrow PV^{\langle 1 \rangle} \\ \Delta \leftarrow \sqrt{r\cdot r} \\ r - 0.00578\cdot\Delta\cdot v \end{array} \right.$$

We now define **U2PV** as follows.

$$U2PV(K, q, e, i, \Omega, \omega, \Delta t) := \begin{array}{|l} \alpha \leftarrow K^2 \cdot \dfrac{(1-e)}{q} \\ p \leftarrow q \cdot (1+e) \\ s \leftarrow \dfrac{\Delta t}{q} \\ \Delta s \leftarrow s \\ \text{while } |\Delta s| \geq 0.00000001 \\ \quad \begin{array}{|l} c \leftarrow C\left(\alpha \cdot s^2\right) \\ f \leftarrow q \cdot s + K^2 \cdot e \cdot s^3 \cdot c_3 - \Delta t \\ Df \leftarrow q + K^2 \cdot e \cdot s^2 \cdot c_2 \\ DDf \leftarrow K^2 \cdot e \cdot s \cdot c_1 \\ \text{if } Df \geq 0 \\ \quad \| \; m \leftarrow 1 \\ \text{else} \\ \quad \| \; m \leftarrow -1 \\ \Delta s \leftarrow \dfrac{-5 \cdot f}{\left(Df + m \cdot \sqrt{\left|(4 \cdot Df)^2 - 20 \cdot f \cdot DDf\right|}\right)} \\ s \leftarrow s + \Delta s \end{array} \\ rcosv \leftarrow q - K^2 \cdot s^2 \cdot c_2 \\ rsinv \leftarrow K \cdot \sqrt{p} \cdot s \cdot c_1 \\ r \leftarrow PQEQ(i, \Omega, \omega, rcosv, rsinv) \\ a \leftarrow \dfrac{-K}{\sqrt{p}} \cdot \dfrac{rsinv}{Df} \\ b \leftarrow \dfrac{K}{\sqrt{p}} \cdot \left(e + \dfrac{rcosv}{Df}\right) \\ v \leftarrow PQEQ(i, \Omega, \omega, a, b) \\ \text{augment}(r, v) \end{array}$$

Note that the output of function **U2PV** is a 3x2 matrix containing position and velocity. Function **FCOM** will assign the output of **U2PV** to the 3x2 matrix **PV**, and will then apply function **LTIM** to **PV**, in order to effect the light-time correction, as follows.

$$FCOM(q, e, i, \Omega, \omega, T, JD, \Delta d, N) := \begin{array}{l} \text{for } j \in 0..N \\ \quad \Delta t \leftarrow JD - T \\ \quad K \leftarrow k \\ \quad PV \leftarrow U2PV(K, q, e, i, \Omega, \omega, \Delta t) \\ \quad PV^{\langle 0 \rangle} \leftarrow PV^{\langle 0 \rangle} - HGEO(JD) \\ \quad r \leftarrow LTIM(PV) \\ \quad r \leftarrow ECEQ(r) \\ \quad \alpha \leftarrow \text{angle}(r_0, r_1) \cdot \dfrac{DegPerRad}{15} \\ \quad \delta \leftarrow \text{asin}\left(\dfrac{r_2}{\sqrt{(r_0)^2 + (r_1)^2 + (r_2)^2}}\right) \cdot DegPerRad \\ \quad \text{if } j = 0 \\ \quad\quad Table \leftarrow [JD \;\; \alpha \;\; \delta] \\ \quad \text{else} \\ \quad\quad Table \leftarrow \text{stack}(Table, [JD \;\; \alpha \;\; \delta]) \\ \quad JD \leftarrow JD + \Delta d \\ Table \end{array}$$

12. Generate formatted ephemeris table for Comet Hale-Bopp.

(To generate predictions for your own comet or asteroid, for your own time period of interest, simply change the following ten numbered input quantities to what you wish. You can obtain current orbital elements for comets and asteroids from the Minor Planet Center [1].)

$q := 0.9143839$ **Input 1**. Perihelion distance, A.U.

$e := 0.9952982$ **Input 2**. Orbital eccentricity.

$i := \dfrac{089.43088}{DegPerRad}$ **Input 3**. Orbital inclination, degrees.

$\Omega := \dfrac{282.47058}{DegPerRad}$ **Input 4**. Celestial longitude of ascending node, degrees.

$\omega := \dfrac{130.56797}{DegPerRad}$ **Input 5**. Argument of perihelion, degrees.

$JD_o := 2450448.5$ **Input 6**. Julian date for 1997 Jan 0.0 TT.

$T := JD_o + 90.95962$ **Input 7**. Julian date of perihelion passage.

$JD := JD_o + 76$ **Input 8**. Julian date of ephemeris start, 1997 March 17.0 TT.

$\Delta d := 5.0$ **Input 9**. Ephemeris step size, days.

$N := 12$ **Input 10**. Number of time steps to take.

$M := FCOM(q, e, i, \Omega, \omega, T, JD, \Delta d, N)$

$FORM(M, N) =$

Julian Date	R.A. hr	mm	ss.s	Dec. dg	mm	ss	Ephemeris Matrix Month & Day, 1997:
2450524.5	23	21	10.7	43	59	41	Mar 17
2450529.5	0	9	57.9	45	33	14	22
2450534.5	1	0	21.8	45	42	41	27
2450539.5	1	48	23.2	44	29	32	Apr 1
2450544.5	2	31	4.1	42	11	5	6
2450549.5	3	7	18.5	39	11	8	11
2450554.5	3	37	26.1	35	51	7	16
2450559.5	4	2	25.2	32	26	17	21
2450564.5	4	23	19.8	29	5	53	26
2450569.5	4	41	6.1	25	54	44	May 1
2450574.5	4	56	28.5	22	54	43	6
2450579.5	5	10	0.6	20	6	5	11
						⋮	16

When we compare the output with that from the Montenbruck & Pfleger **MPF** ephemeris matrix (refer again to Step 11), we now see differences in right ascension no larger than about 0.7 seconds of time, and differences in declination no larger than about 1 second of arc, for approximately a factor of three improvement in R.A. agreement, and a factor of nine improvement in DEC agreement. Better agreement is not possible without including perturbations of the Earth-moon system by the other planets of the solar system, as is done in Montenbruck & Pfleger's COMET program.

NOTES AND COMMENTS

1. The first four c-functions are calculated by series and recursion, by the algorithm of Danby [2, Section 6.9]. Stiefel and Scheifele [9, p. 43] have named these functions the *Stumpff functions* in honor of Karl J. Stumpff. Stumpff's key reference to these functions, in which he shows that they form the basis for a representation of two-body motion which is the same (uniform) for all path eccentricities (i.e., the same for circular, elliptical, parabolic, and hyperbolic two-body paths), is Reference 10, below. [Stumpff also published, in German, a trilogy of celestial mechanics, *Himmelsmechanik* (volumes in 1959, 1965, and 1974).]

2. The word "uniform" is used in the sense of Note 1. Here k is the Gaussian gravity constant for the sun, as the primary in a two-body system consisting of the sun and the comet. Function **U2PM** is also used with the sun as the primary and the Earth-moon barycenter as the secondary, in Steps 4 and 7). The inputs to **U2PM** are

K	function of k and μ (see Mathcad "UPM Notation" document)
q	perihelion distance, in A.U.
e	orbital eccentricity (e = 0 for circle, 0<e<1 for ellipse, e=1 for parabola, and e>1 for hyperbola)
i	orbital inclination, in radians
Ω	celestial longitude of ascending node, in radians
ω	argument of perihelion, in radians
Δt	time of flight from perihelion to time of interest

The **U2PM** function closely follows the treatment by Mansfield [3], except that the *while* loop iterates on the fictitious time, s, by a second-order root-finding method called *the algorithm of Laguerre-Conway* by Danby [2, p. 160]; see also Prussing and Conway [7, p. 38].

3. The data for the orbit of comet Hale-Bopp are as calculated by Brian G. Marsden, director of the Minor Planet Center, Smithsonian Astrophysical Observatory [4]. Later, more accurate orbital elements are available, but I chose these to make a point: excellent orbital elements for Hale-Bopp were available more than eighteen months before perihelion.

4. This step demonstrates how uniform two-body mechanics, in the form of **U2PM**, can be applied to perturbed orbits, such as the orbit of the Earth-moon barycenter around the sun. First, the orbital elements are updated for secular perturbations, as per the algorithm given by Seidelmann [8], then **U2PM** is used with the updated elements to calculate position. Note that μ is not taken as unity, since the mass of the Earth-moon barycenter, in solar masses, is measurable. Also, the position coordinates are referred to the mean equator and equinox of the epoch J2000.0, which is 2000 January 1.5 TT. [A note on notations for time: it is customary now to use the notation UT for universal time when the exact kind (UT1, UT2, or UTC) is not important to the discussion. Similarly, TT is used to denote terrestrial time (TDT, TDB, TDC). TT replaces ET (ephemeris time) in current usage.]

5. The algorithm for correcting the position of the Earth-moon barycenter to the geocenter is given in the *Astronomical Almanac* [6], and requires calculating the mean orbital longitude of the moon as a function of the Julian centuries elapsed since J2000.0 (see again Note 4).

6. Given the heliocentric ecliptic position of the geocenter and the heliocentric ecliptic position of the comet, we simply subtract the first vector from the second to get the geocentric ecliptic position of the comet. We then use the obliquity of the ecliptic, ε, as the argument of the rotation matrix **M**, which transforms geocentric ecliptic position to geocentric equatorial position. Finally, we calculate the spherical polar angular coordinates α and δ from the cartesian positional coordinates.

7. The function **HGEO** converts Steps 4 and 5, which calculate a single heliocentric ecliptic position of the geocenter, into a procedure which can be embedded into an ephemeris generation function.

8. Function **ECEQ** converts the matrix definition and matrix multiplication in Step 6 into a procedure which can be used to transform from geocentric ecliptic to geocentric equatorial.

9. **ECOM** is our first ephemeris generation function. It produces a table with rows consisting of Julian date, right ascension, and declination. There are as many rows as ephemeris points requested. T is the time of perihelion passage, JD is the Julian date for the first ephemeris point, Δd is the time step, in days, and N is the number of ephemeris points requested.

10. Here **ECOM** is used to generate an ephemeris for comet Hale-Bopp. The output gives right ascension in hours and fractional hours, and gives declination in degrees and fractional degrees. For comparison with ephemerides produced from other programs, it is desirable to express right ascensions in hours, minutes, seconds, and tenths of seconds of time, and declinations in degrees, minutes, and seconds of arc. We do this by defining and applying the function **FORM** in the next step, below, but first we place the output of **ECOM** into the ephemeris matrix **M**.

11. Here we define a function, **FORM**, that formats the right ascensions and declinations of the ephemeris. It is a straightforward application of the Mathcad *floor* and *augment* functions. But it should be noted that **FORM** rounds right ascension to the nearest tenth of a second of time, and rounds declination to the nearest second of arc. In this step we also discover that an **ECOM**-generated ephemeris of a comet lacks the light-time correction, and correcting for light-time requires that we know the velocity of the comet as well as its position. Following definition of the functions **PQEQ** and **LTIM**, we are able to define the uniform, two-body mechanics path propagation function **U2PV**, which calculates velocity as well as position, and the ephemeris generating function **FCOM**, which generates a comet ephemeris in which the positions are corrected for light-time.

12. In this step we use **FCOM** to generate a comet ephemeris in which positions are corrected for the time it takes light to travel from the comet to Earth. The inputs are clearly labeled and numbered to facilitate adapting Step 12 to the generation of an ephemeris for any comet of interest, for any time period of interest. It should be noted, however, that the ephemeris is a "two-body ephemeris". In practical terms, this means that the epoch of the orbital elements should be near the time period of interest. Perturbed comet ephemerides can be generated by techniques given in Montenbruck & Pfleger [5, Chapter 5].

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UNIFORM PATH MECHANICS (UPM) NOTATIONAL SUMMARY

N	a counter variable that starts at zero
x	argument of Stumpff's c-functions
c	a vector with Stumpff's first four c-functions as components
k	Gaussian constant for primary in system of two gravitating bodies, or "two-body system"
μ	1 + m, where m is the mass of the secondary body in the two-body system, in units of the primary body's mass
K	$k \cdot \sqrt{\mu}$ [a notation adopted by Stiefel and Scheifele in *Linear and Regular Celestial Mechanics* (1971)]
q	periapsis distance of two-body trajectory, e.g., perihelion distance in astronomical units (A.U.) or perigee distance in Earth radii (E.R.)
e	orbital eccentricity, a measure of the shape of a two-body trajectory
i	orbital inclination, i.e., the angle that a comet's orbital plane makes with the ecliptic plane, or the angle that an Earth satellite's orbital plane makes with Earth's equatorial plane
Ω	reference angle of ascending node, e.g., *celestial longitude* of the ascending node of a comet's orbit, and *right ascension* of the ascending node of an Earth satellite's orbit
ω	argument of periapsis, i.e., argument of perihelion of a comet's orbit, or argument of perigee of an Earth satellite's orbit
Δt	time of flight from periapsis (perihelion or perigee) to the epoch of the orbital elements
α	twice the negative of the total energy in a two-body system (as used in **U2PM** and **U2PV**); also, the right ascension coordinate (R.A.) of a body on the celestial sphere
p	q (1+e), the semi-latus rectum of a conic path (circle, ellipse, parabola, or hyperbola)

s	"fictitious time" variable (independent variable) of uniform path mechanics (UPM)
f	as used in **U2PM** and **U2PV**, the function f(s) associated with the uniform Kepler equation (note that the derivative of f with respect to s is the radius vector, in accordance with the Sundmann transformation, dt/ds = r)
m	as used in **U2PM** and **U2PV**, a sign variable associated with the Laguerre-Conway second-order root-finding method
Δs	change in, or correction to s (see s, above)
P	a unit vector that points from the primary body's (sun's or Earth's) center to the point of periapsis (perihelion or perigee) of the secondary body's orbit
Q	**W**x**P**, where **W** is the unit orbital angular momentum vector obtained by crossing **r** with **v**, and then unitizing the resulting vector (**P**, **Q**, and **W** are the basis vectors for a dextral, orthonormal orbital reference frame called the *perifocal orbit reference frame*; this is an inertial reference frame under the assumptions of two-body motion)
T	time of periapsis passage, one of the six "conic" orbital elements
r_C	heliocentric ecliptic position vector of comet
JD	Julian date
JD_o	Julian date at some reference epoch, here J2000.0, or 2000 January 1.5 Terrestrial Time (TT)
ΔT	Time elapsed in Julian centuries of 36525.0 days
n	mean orbital motion of secondary (here, the Earth-moon barycenter)
L	heliocentric mean orbital longitude of Earth-moon barycenter
r_{EM}	heliocentric ecliptic position of Earth-moon barycenter
L_M	geocentric mean orbital longitude of moon

r	geocentric position vector of comet, with components in A.U.
ε	obliquity of the ecliptic (i.e., the angle that the sun's ecliptic path on the celestial sphere makes with the celestial equator)
M	a matrix that converts geocentric ecliptic positions to geocentric equatorial positions; also used as a working matrix for storing comet ephemerides
δ	comet's declination (Dec.) coordinate on celestial sphere
Δd	time step for comet ephemeris calculations
v	geocentric velocity vector of comet, with components in A.U./day
Δ	geocentric distance to comet, usually expressed in A.U.
PV	3x2 matrix whose columns are **r** (position) and **v** (velocity)
MPF	matrix of cometary ephemeris points obtained by running Montenbruck & Pfleger's COMET program (see Reference 5)
DegPerRad	number of degrees in one radian
SecPerDeg	number of seconds in one degree
SecPerRev	number of seconds in one orbital revolution of 360 degrees

UPM FUNCTION SUMMARY

C	calculates first four of Stumpff's c-functions
U2PM	calculates comet's, planet's, or Earth satellite's orbital position by uniform, two-body path mechanics
HGEO	calculates heliocentric ecliptic position of Earth's center, referred to the mean eclicptic and equinox of J2000.0
ECEQ	converts geocentric ecliptic coordinates to geocentric equatorial coordinates at the J2000.0 epoch
ECOM	calculates a comet's ephemeris, a table of positions on the celestial sphere at equally-spaced Julian dates (times) of Terrestrial Time (TT), but does not correct positions for light-time
FORM	formats the output of the ephemeris-generating functions ECOM and FCOM
PQEQ	transforms a comet's or Earth satellite's perifocal coordinates (position and velocity, in turn) to heliocentric ecliptic (comet's) or geocentric equatorial (Earth satellite's) coordinates
LTIM	corrects comet's geocentric position vector for light-time, using comet's velocity vector
U2PV	calculates comet's, planet's, or Earth satellite's orbital position and velocity by uniform, two-body path mechanics
FCOM	calculates comet's ephemeris at equally-spaced Julian dates of TT, and corrects the positions for the time it takes light to travel from the comet to Earth (light-time correction)