TWO-BODY ORBIT PROPAGATION VIA STATE SPACE ANALYSIS

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In this worksheet we propagate numerically the Near-Earth Asteroid Rendezvous (NEAR) spacecraft's solved-for Earth escape trajectory [1, p. 5]. We integrate from epoch to the time of the first observation, printing out the initial and final steps and eight steps in between.

Define the Gaussian constant for Earth as primary and then the related constant, K, using the value of k_e adopted by U.S. Space Command for use with its general perturbation theories.

$k_e := 0.074366916133$ Gaussian constant for Earth as primary body.

$m := 0$ Secondary's mass, negligible relative to primary's mass.

$K := k_e \cdot \sqrt{1+m}$

$K = 0.074366916133$ E.R.$^{3/2}$ per minute.

$a_e := 6378.135$ We will also need Earth's equatorial radius to convert from km to Earth radii and vice versa.

Define and propagate numerically the state space equation X_{dot} = S(X) X. Start with position and velocity at epoch and step to position and velocity at the time of the first observation. Note that X_o has units of E.R. for position and E.R./min for velocity.

$$r_o := \begin{bmatrix} -6342.275645 \\ 3691.745453 \\ -3184.242135 \end{bmatrix} \cdot \frac{1}{a_e} \qquad v_o := \begin{bmatrix} -7.849870004 \\ -6.450404087 \\ -4.704817677 \end{bmatrix} \cdot \frac{60}{a_e}$$

$X_o := stack\left(r_o, v_o\right)$ Note that X_o is a 6-by-1 column vector obtained by "stacking" position on top of velocity.

$$S(X) := \left\| \begin{array}{l} s \leftarrow -K^2 \cdot \left(\left(X_0\right)^2 + \left(X_1\right)^2 + \left(X_2\right)^2\right)^{\frac{-3}{2}} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ s & 0 & 0 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & 0 & 0 \end{bmatrix} \end{array} \right.$$

Integrate from t_o = 0 (epoch) to t (time of first observation). To facilitate comparison with the BASIC program, use fixed-step Runge-Kutta integration with step size of 4.2052583 minutes.

$t_o := 0.0$ — Times t_o and t are in minutes.

$t := 59 + \frac{32.510}{60} - \left(17 + \frac{29.355}{60}\right)$ — $t = 42.052583$

$NRK := 10$ — Tell Mathcad's rkfixed function to output 10 Runge-Kutta integration points.

$D\left(t, X_o\right) := S\left(X_o\right) \cdot X_o$ — Tell Mathcad's rkfixed function what the derivative function is.

$Z := rkfixed\left(X_o, t_o, t, NRK, D\right)$ — Invoke rkfixed to integrate X_o from time t_o to time t and output NRK ephemeris points.

$$Z = \begin{bmatrix} 0.000000 & -0.994378 & 0.578813 & -0.499243 & -0.073845 & -0.060680 & -0.044259 \\ 4.205258 & -1.281944 & 0.313080 & -0.673636 & -0.063349 & -0.064938 & -0.038861 \\ 8.410517 & -1.530867 & 0.037055 & -0.827781 & -0.055441 & -0.065993 & -0.034645 \\ 12.615775 & -1.751167 & -0.240075 & -0.966478 & -0.049628 & -0.065668 & -0.031469 \\ 16.821033 & -1.950329 & -0.514513 & -1.093523 & -0.045297 & -0.064803 & -0.029060 \\ 21.026292 & -2.133569 & -0.784862 & -1.211649 & -0.041993 & -0.063761 & -0.027196 \\ 25.231550 & -2.304518 & -1.050754 & -1.322798 & -0.039410 & -0.062699 & -0.025721 \\ 29.436808 & -2.465748 & -1.312255 & -1.428372 & -0.037344 & -0.061679 & -0.024530 \\ 33.642067 & -2.619129 & -1.569602 & -1.529401 & -0.035658 & -0.060726 & -0.023549 \\ 37.847325 & -2.766052 & -1.823096 & -1.626659 & -0.034259 & -0.059846 & -0.022730 \\ 42.052583 & -2.907573 & -2.073042 & -1.720743 & -0.033080 & -0.059038 & -0.022034 \end{bmatrix}$$

And so we obtain the following values of position in km and velocity in km/sec at time t:

$$\begin{bmatrix} Z_{NRK,1} \\ Z_{NRK,2} \\ Z_{NRK,3} \end{bmatrix} \cdot a_e = \begin{bmatrix} -18544.895 \\ -13222.14 \\ -10975.129 \end{bmatrix} \quad \text{(km)} \qquad \begin{bmatrix} Z_{NRK,4} \\ Z_{NRK,5} \\ Z_{NRK,6} \end{bmatrix} \cdot \frac{a_e}{60} = \begin{bmatrix} -3.5165086 \\ -6.2758761 \\ -2.3422877 \end{bmatrix} \quad \text{(km/sec)}$$

REFERENCE

[1] Mansfield, Roger L., "Tracking the NEAR Launch," The Orrery, issue #10 (April 1996).

COMMENTS

1. Note that we can integrate a perturbed trajectory as X_{dot} = S(X) X + P(X) if we know P(X), a six-vector whose first three components are null and whose last three components are the perturbative accelerations in the ECI x, y, and z directions.

Further, we can find the state transition matrix at each observation time by integrating Φ_{dot} = A(X) Φ, where A is a matrix of partial derivatives found by differentiating the perturbative acceleration terms.

2. For further information about these concepts and how they can be applied to the orbit determination process, check out the website for Prof. George H. Born's CU-Boulder Aerospace Engineering course ASEN 5070, Introduction to Statistical Orbit Determination, at

http://www-ccar.colorado.edu/~goldstdb/ASEN5070.htm.