THE EFFECT OF A RADIAL IMPULSE

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We consider the effect of a purely radial impulse on the orbital elements of an artificial Earth satellite in a circular orbit. First we define some constants and conventions that we will use in this worksheet.

$DegPerRad := \frac{180}{\pi}$ Number of degrees in one radian.

$k_e := 0.07436691613$ Gaussian constant for Earth-relative motion.

$\mu := 1$ The secondary's mass is negligible relative to Earth's mass.

$a_e := 6378.135$ Earth's mean equatorial radius, in km.

$K := k_e \cdot \sqrt{\mu}$ $\mathbf{ORIGIN} \equiv 1$ (ORIGIN=1 forces all vector and matrix subscripts to start at one.)

$KMSEC := a_e \cdot \frac{K}{60}$ Conversion from E.R./kemin to km/sec.

We can determine experimentally what will happen if we apply a 1.0 km/sec outward, radial impulse to a geostationary orbit by the following procedure:

1. Construct a state vector (position and velocity) for the geostationary orbital radius a = 35786 + 6378.135 km = 42164.135 km. Transform this state vector to orbital elements.

2. Add a 1.0 km/sec outward, radial impulse to the velocity vector. Transform this second state vector to orbital elements and note the differences between the two orbits.

We find that:

a. The perfectly circular orbit becomes elliptical. The point at which the impulse is applied becomes one of the two points in the orbit that are the endpoints of the latus rectum.

b. Apogee occurs 90 degrees from the point of the impulse (υ = 90 degrees), and perigee occurs 270 degrees from the point of the impulse (υ = 270 degrees).

Now let's prove these assertions by analysis. Let's start with a perfectly circular orbit at geosynchronous altitude and let us define ΔV as the impulsive radial velocity change. Then

$r := \frac{42164.135}{a_e}$ $\Delta V := \frac{1.0}{KMSEC}$ (E.R./kemin)

$V := \frac{1}{\sqrt{r}}$ $V \cdot KMSEC = 3.07466$ (km./sec)

Note that ΔV and V have units of E.R./kemin. We now apply ΔV to V. The new velocity vector magnitude, V_1, is given by

$V_1 := \sqrt{V^2 + \Delta V^2}$ since ΔV is purely in the radial direction.

The flight path angle, which before was zero, is now

$\phi := \operatorname{asin}\left(\frac{\Delta V}{V_1}\right)$ $\phi \cdot DegPerRad = 18.01654$ (degrees)

The new angular momentum vector magnitude, h, is (in canonical units)

$h := r \cdot V_1 \cdot \cos(\phi)$ $h = 2.57113$

The new specific mechanical energy is (in canonical units)

$E := \frac{{V_1}^2}{2} - \frac{1}{r}$ $E = -0.06763$

So the new orbital eccentricity is

$ecc := \sqrt{1 + 2 \cdot E \cdot h^2}$ $ecc = 0.32524$

The new semimajor axis is

$a := \frac{-1}{2 \cdot E}$ $a \cdot a_e = 47151.87744$ (km)

The new radii of apogee and perigee are

$r_a := a \cdot (1 + ecc)$ $\quad$ $r_a \cdot a_e = 62487.50329$ $\quad$ (km)

$r_p := a \cdot (1 - ecc)$ $\quad$ $r_p \cdot a_e = 31816.25159$ $\quad$ (km)

We must show that the argument of perigee is rotated 270 degrees from the impulse point. Note that before the radial impulse, we were at perigee. If perigee were shifted by 270 degrees, then we should have a true anomaly υ of 90 degrees after the radial impulse. Now $r=p/(1 + e\cos\upsilon)$, so

$p := h^2$ $\quad$ (again, when h is in canonical units)

$\upsilon := \operatorname{acos}\left(\frac{1}{ecc} \cdot \left(\frac{p}{r} - 1\right)\right)$ $\quad$ (note that p = r, making cosine = 0)

$\upsilon \cdot DegPerRad = 90$ $\quad$ (degrees).

The period of the orbit was about 1436 minutes before the radial impulse. After the radial impulse, it is

$P := \frac{2 \cdot \pi}{K \cdot a^{\frac{-3}{2}}}$ $\quad$ $P = 1698.27501$ $\quad$ (minutes)

What happens if the radial impulse is directed toward the geocenter, rather than away from it? In this case, we get the same orbital shape, but apogee and perigee are switched, i.e., the figure is flipped about the latus rectum.

We cannot arrive at this result merely by changing the sign of ΔV above, since $\cos\upsilon$ is the same for both positive and negative values of υ. But we can use a symmetry argument: if we draw the V_1 that corresponds to $-\Delta V$ and flip the figure about the latus rectum, we see that the flipped velocity vector coincides with the velocity vector that we would have if we were traveling along the orbit in the opposite direction, and headed toward perigee, after traveling only 90 degrees of true anomaly. That is, we would be at a true anomaly of 270 degrees. So our true anomaly for the unflipped post-impulse orbit must be 270 degrees when the impulse is directed toward the geocenter, making the argument of perigee ω = 90 degrees.

We close with a Mathcad plot that shows the orbits before and after an outward radial impulse of 1.0 km/sec.

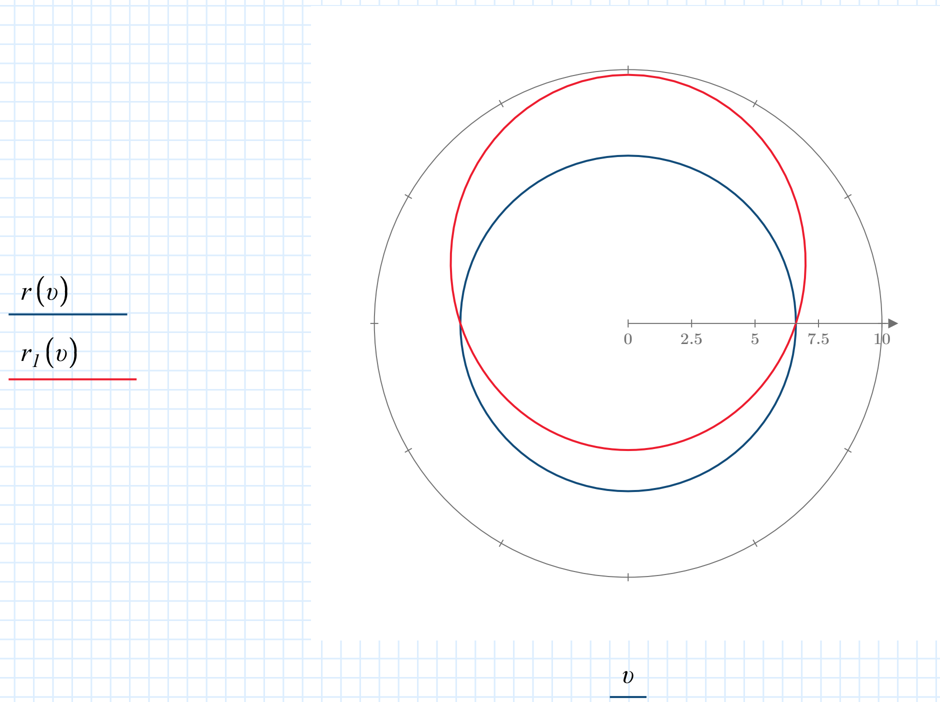
$p := r$

$p_1 := h^2$

$r(\upsilon) := p$

$$r_1(\upsilon) := \frac{p}{1 + ecc \cdot \cos\left(\upsilon - \frac{3\ \pi}{2}\right)}$$

$$\upsilon := 0, 2 \cdot \frac{\pi}{100} .. 2 \cdot \pi$$



Note that trace 1 (blue) is the original orbit and trace 2 (red) is the orbit after an outward, radial 1.0 km/sec impulse has been applied at x = 6.61073 E.R. and y = 0. Finally, note that if we make the argument of the cosine in the formula for r_1 equal to $\upsilon - \pi/2$, we get the post-maneuver orbit that corresponds to an inward, radial impulse of 1.0 km/sec (try it!).

This completes our analysis of the effect of a radial impulse on a circular orbit.