GAUSS'S ANGLES-ONLY METHOD WITH UPM ENHANCEMENTS:
APPLICATION TO THE ORBIT OF ASTEROID 1997 XF_{11}

Roger L. Mansfield, June 5, 2001
http://astroger.com

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Gauss's angles-only orbit determination method uses three (the minimum number) of angles-only observations to determine a preliminary, two-body solution that fits exactly the three observations at hand. Gauss first published his method in 1809 [1]. Variations on the original angles-only method have been provided in several textbooks; see, for example, Escobal [2] and Danby [3]. In what follows I will work with Escobal's exposition, and will use the Uniform Path Mechanics (UPM) theory to extend Gauss's angles-only method, using equations published in [4] and [5]. I should note that by the term "UPM theory" I mean the representation of two-body paths (position, velocity, and state transition matrix) of arbitrary eccentricity by means of f and g functions of Stumpff's c-functions, as treated by Stumpff [6], Stiefel and Scheifele [7], Goodyear [8], Danby [9] and Mansfield [10].

Escobal provides a concise exposition of Gauss's angles-only method in essentially two forms: the original method, with iteration on the three possible area ratios of sector to triangle, and a variation that uses the so-called Herrick-Gibbs modification (an orbit determination method in its own right, given three positional observations) to provide an approximate solution based upon expanding the f and g functions in time series. The original Gaussian method, as presented by Escobal, is limited to elliptical orbits. It is certainly possible to construct variants of Gauss's method that use algorithms for parabolic and hyperbolic paths, algorithms which closely parallel the elliptical orbit algorithm. But the UPM approach that I present below is to be preferred precisely because it is uniform: "one form" suffices for all three possible path regimes: elliptical, parabolic, and hyperbolic.

The Herrick-Gibbs modification improves upon the original Gaussian scheme by providing a truncated Taylor time series representation of position that works for any path eccentricity. It is a "uniform" improvement to the original Gaussian scheme and is very effective. However, the representation is not exact because the Taylor time series, one for each positional component, together yield a path that only approximates the true two-body path. By expressing Gauss's hypergeometric X function as a quotient of c-functions, one makes the sector-to-triangle area ratio iteration into a uniform algorithm. In so doing, one extends Gauss's original angles-only method so that it becomes an exact two-body orbit determination scheme that has the same form for elliptical, parabolic, and hyperbolic paths.

This worksheet will determine the orbit of the minor planet 1997 XF_{11} using three actual observations from Minor Planet Electronic Circular (MPEC) 1997-Y11, dated 1997 December 23. The steps are:

1. Specify the three RA/DEC observations relative to the ECI equatorial J2000.0 frame.

2. Specify the sun's location, **R**, in ECI equatorial coordinates at the three observation times.

3. Convert the RA/DEC observations into three ECI equatorial J2000.0 direction vectors, **L**.

4. Compute first estimates of the geocentric distances ρ_i (i = 1, 2, 3).

5. Iterate on the ρ_i with sub-iterations on the three possible area ratios y_{12}, y_{23}, and y_{13}.

6. Upon convergence of all iterations, calculate $\mathbf{r}_2$ and $\mathbf{v}_2$ in HCI equatorial J2000.0 coordinates.

The final two steps, though not a part of Gauss's angles-only method, are

7. Transform $\mathbf{r}_2$ and $\mathbf{v}_2$ from the HCI equatorial J2000.0 frame to the HCI ecliptic J2000.0 frame.

8. Transform $\mathbf{r}_2$ and $\mathbf{v}_2$ to conic elements.

1. Specify the three RA/DEC observations relative to the ECI equatorial J2000.0 reference frame.

$$DegPerRad := \frac{180}{\pi}$$

Set Mathcad ORIGIN to 1 so that vector subscripts start with 1 rather than with zero.

$$SecPerDeg := 3600.0$$

$$\mathbf{ORIGIN} \equiv 1$$

$$SecPerRev := SecPerDeg \cdot 360.0$$

$$t_1 := 2450788.5 + 0.47227 \qquad t_1 = 2450788.97227000$$

1997 Dec 06.47227

$$t_2 := 2450800.5 + 0.69766 \qquad t_2 = 2450801.19766000$$

1997 Dec 18.69766

$$t_3 := 2450803.5 + 0.65311 \qquad t_3 = 2450804.15311000$$

1997 Dec 21.65311

$$DEC_1 := \frac{13 + \frac{31.27167}{60}}{DegPerRad} \qquad DEC_1 = 0.23598936$$

$$DEC_2 := \frac{13 + \frac{42.03833}{60}}{DegPerRad} \qquad DEC_2 = 0.23912126$$

$$DEC_3 := \frac{13 + \frac{48.18167}{60}}{DegPerRad} \qquad DEC_3 = 0.24090828$$

$$RA_1 := \frac{15 \cdot \left(07 + \frac{58.49583}{60}\right)}{DegPerRad} \qquad RA_1 = 2.08783192$$

$$RA_2 := \frac{15 \cdot \left(07 + \frac{38.23883}{60}\right)}{DegPerRad} \qquad RA_2 = 1.99944409$$

$$RA_3 := \frac{15 \cdot \left(07 + \frac{32.44667}{60}\right)}{DegPerRad} \qquad RA_3 = 1.97417102$$

2. Specify the sun's location, **R**, in ECI equatorial J2000.0 coordinates at the three observation times.

We will need function **C** to calculate the first four c-functions for function **U2PM**.

$$
\begin{aligned}
C(x) := \;& N \leftarrow 0 \\
& \text{while } |x| \geq 0.1 \\
& \quad x \leftarrow \frac{x}{4} \\
& \quad N \leftarrow N + 1 \\
& c_3 \leftarrow \frac{\left(1 - \frac{x}{20} \cdot \left(1 - \frac{x}{42} \cdot \left(1 - \frac{x}{72} \cdot \left(1 - \frac{x}{110} \cdot \left(1 - \frac{x}{156} \cdot \left(1 - \frac{x}{210}\right)\right)\right)\right)\right)\right)}{6} \\
& c_2 \leftarrow \frac{\left(1 - \frac{x}{12} \cdot \left(1 - \frac{x}{30} \cdot \left(1 - \frac{x}{56} \cdot \left(1 - \frac{x}{090} \cdot \left(1 - \frac{x}{132} \cdot \left(1 - \frac{x}{182}\right)\right)\right)\right)\right)\right)}{2} \\
& c_1 \leftarrow 1 - c_3 \cdot x \\
& c_0 \leftarrow 1 - c_2 \cdot x \\
& \text{while } N > 0 \\
& \quad N \leftarrow N - 1 \\
& \quad c_3 \leftarrow \frac{(c_1 \cdot c_2 + c_3)}{4} \\
& \quad c_2 \leftarrow \frac{c_1 \cdot c_1}{2} \\
& \quad c_1 \leftarrow c_1 \cdot c_0 \\
& \quad c_0 \leftarrow 2 \cdot c_0 \cdot c_0 - 1 \\
& \left[c_0 \;\; c_1 \;\; c_2 \;\; c_3\right]^{T}
\end{aligned}
$$

We will need the uniform, two-body path propagator function, **U2PM**, which will be invoked by function **HGEO**.

$k := 0.01720209895$ $\mu := 1.0$

$K := k \cdot \sqrt{\mu}$

$$
\begin{aligned}
U2PM(K, q, e, i, \Omega, \omega, \Delta t) := \;& \alpha \leftarrow K^2 \cdot \frac{(1-e)}{q} \\
& p \leftarrow q \cdot (1+e) \\
& s \leftarrow \frac{\Delta t}{q} \\
& \Delta s \leftarrow s \\
& \text{while } |\Delta s| \geq 0.00000001 \\
& \quad c \leftarrow C(\alpha \cdot s^2) \\
& \quad f \leftarrow q \cdot s + K^2 \cdot e \cdot s^3 \cdot c_4 - \Delta t \\
& \quad Df \leftarrow q + K^2 \cdot e \cdot s^2 \cdot c_3 \\
& \quad DDf \leftarrow K^2 \cdot e \cdot s \cdot c_2 \\
& \quad \text{if } Df \geq 0 \\
& \quad \quad m \leftarrow 1 \\
& \quad \text{else} \\
& \quad \quad m \leftarrow -1 \\
& \quad \Delta s \leftarrow \frac{-5 \cdot f}{\left(Df + m \cdot \sqrt{\left|(4 \cdot Df)^2 - 20 \cdot f \cdot DDf\right|}\right)} \\
& \quad s \leftarrow s + \Delta s \\
& P_1 \leftarrow \cos(\Omega) \cdot \cos(\omega) - \sin(\Omega) \cdot \cos(i) \cdot \sin(\omega) \\
& P_2 \leftarrow \sin(\Omega) \cdot \cos(\omega) + \cos(\Omega) \cdot \cos(i) \cdot \sin(\omega) \\
& P_3 \leftarrow \sin(i) \cdot \sin(\omega) \\
& Q_1 \leftarrow -(\cos(\Omega) \cdot \sin(\omega) + \sin(\Omega) \cdot \cos(i) \cdot \cos(\omega)) \\
& Q_2 \leftarrow -(\sin(\Omega) \cdot \sin(\omega) - \cos(\Omega) \cdot \cos(i) \cdot \cos(\omega)) \\
& Q_3 \leftarrow \sin(i) \cdot \cos(\omega) \\
& c \leftarrow C(\alpha \cdot s^2) \\
& rcosv \leftarrow q - K^2 \cdot s^2 \cdot c_3 \\
& rsinv \leftarrow K \cdot \sqrt{p} \cdot s \cdot c_2 \\
& rcosv \cdot P + rsinv \cdot Q
\end{aligned}
$$

We use function **HGEO**, which calculates the heliocentric ecliptic J2000.0 coordinates of the geocenter, to calculate the ECI equatorial J2000.0 coordinates of the sun.

$$
HGEO(JD) := \left\|
\begin{array}{l}
JD_o \leftarrow 2451545.0 \\
T_c \leftarrow \dfrac{JD - JD_o}{36525.0} \\
a \leftarrow 1.00000011 - 0.00000005 \cdot T_c \\
e \leftarrow 0.01671022 - 0.00003804 \cdot T_c \\
q \leftarrow a \cdot (1 - e) \\
\mu \leftarrow 1.00000304 \\
K \leftarrow k \cdot \sqrt{\mu} \\
n \leftarrow K \cdot a^{\frac{-3}{2}} \\
\omega \leftarrow \dfrac{102.94719 + \dfrac{1198.28 \cdot T_c}{SecPerDeg}}{DegPerRad} \\
i \leftarrow \dfrac{0.00005 - \dfrac{46.94 \cdot T_c}{SecPerDeg}}{DegPerRad} \\
\Omega \leftarrow 0.0 \\
L \leftarrow \dfrac{100.46435 + \dfrac{1293740.63 + 99 \cdot SecPerRev}{SecPerDeg} \cdot T_c}{DegPerRad} \\
T \leftarrow JD - \dfrac{\mathrm{mod}(L - \omega, 2 \cdot \pi)}{n} \\
\Delta t \leftarrow JD - T \\
r_{EM} \leftarrow U2PM(K, q, e, i, \Omega, \omega, \Delta t) \\
L_M \leftarrow \dfrac{\mathrm{mod}(218.0 + 481268.0 \cdot T_c, 360.0)}{DegPerRad} \\
\begin{bmatrix} r_{EM_1} - 0.0000312 \cdot \cos(L_M) \\ r_{EM_2} - 0.0000312 \cdot \sin(L_M) \\ r_{EM_3} \end{bmatrix}
\end{array}
\right.
$$

We will need the obliquity of the ecliptic, ε, at J2000.0, in order to transform the HCI ecliptic J2000.0 coordinates of Earth to HCI equatorial J2000.0 coordinates.

$$\varepsilon := \frac{23.4392911}{DegPerRad} \qquad M := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\varepsilon) & -\sin(\varepsilon) \\ 0 & \sin(\varepsilon) & \cos(\varepsilon) \end{bmatrix}$$

$ECEQ(r) := M \cdot r$ (Transforms from ecliptic to equatorial.)

$EQEC(r) := M^{-1} \cdot r$ (Transforms from equatorial to ecliptic.)

SUNPOS computes three HCI equatorial J2000.0 positions of Earth, $\mathbf{R}^{\langle i \rangle}$, then converts them to geocentric equatorial J2000.0 positions of the sun, merely by multiplying by -1.

$$SUNPOS := \left\| \begin{array}{l} \text{for } i \in 1..3 \\ \quad \left\| R^{\langle i \rangle} \leftarrow M \cdot HGEO\left(t_i\right) \right. \\ -R \end{array} \right.$$

(**SUNPOS** defined.)

$R := SUNPOS$ (**SUNPOS** invoked.)

$$R = \begin{bmatrix} -0.26472805 & -0.05423869 & -0.00259867 \\ -0.87071490 & -0.90133899 & -0.90252852 \\ -0.37750688 & -0.39078417 & -0.39129989 \end{bmatrix}$$

(**SUNPOS** results.)

(These values agree well with the Astronomical Almanac for 1997, p. C23.)

3. Convert the RA/DEC observations into three ECI equatorial J2000.0 direction vectors, **L**.

$$L^{\langle 1 \rangle} := \begin{bmatrix} \cos\left(DEC_1\right) \cdot \cos\left(RA_1\right) \\ \cos\left(DEC_1\right) \cdot \sin\left(RA_1\right) \\ \sin\left(DEC_1\right) \end{bmatrix}$$

$$L^{\langle 2 \rangle} := \begin{bmatrix} \cos\left(DEC_2\right) \cdot \cos\left(RA_2\right) \\ \cos\left(DEC_2\right) \cdot \sin\left(RA_2\right) \\ \sin\left(DEC_2\right) \end{bmatrix}$$

$$L^{\langle 3 \rangle} := \begin{bmatrix} \cos\left(DEC_3\right) \cdot \cos\left(RA_3\right) \\ \cos\left(DEC_3\right) \cdot \sin\left(RA_3\right) \\ \sin\left(DEC_3\right) \end{bmatrix}$$

$$L = \begin{bmatrix} -0.48060498 & -0.40381482 & -0.38118897 \\ 0.84519469 & 0.88364934 & 0.89318099 \\ 0.23380504 & 0.23684898 & 0.23858478 \end{bmatrix}$$

Compute the angle, in degrees, between observations 1 and 3 for reference. (If the angular spread is too large, the iteration for the ρ_i, as formulated below, will not converge.)

$$\operatorname{acos}\left(L^{\langle 1 \rangle} \cdot L^{\langle 3 \rangle}\right) \cdot DegPerRad = 6.33410354$$

4. Compute first estimates of the geocentric distances ρ_i (i = 1, 2, 3).

The key idea of Gauss's angles-only method is that we can write $\mathbf{r}_2 = c_1\mathbf{r}_1 + c_3\mathbf{r}_3$, where c_1 and c_3 are coefficients to be determined, since the two-body (sun and asteroid) orbital motion must take place in the plane defined by $\mathbf{r}_1 \times \mathbf{r}_3$. We should note that the quantities c_1 and c_3, as used here, are not c-functions; they are simply a notation employed for convenience by Escobal [2].

Since $\mathbf{r} = \rho - \mathbf{R}$, we can write $c_1(\rho_1 - \mathbf{R}_1) + c_3(\rho_3 - \mathbf{R}_3) - (\rho_2 - \mathbf{R}_2) = \mathbf{0}$, or

$c_1\rho_1 + c_2\rho_2 + c_3\rho_3 = c_1\mathbf{R}_1 + c_2\mathbf{R}_2 + c_3\mathbf{R}_3$, where $c_2 = -1$.

Since $\rho = \rho\mathbf{L}$, we can write $c_1\rho_1\mathbf{L}_1 + c_2\rho_2\mathbf{L}_2 + c_3\rho_3\mathbf{L}_3 = \mathbf{G}$, where

$\mathbf{G} = c_1\mathbf{R}_1 + c_2\mathbf{R}_2 + c_3\mathbf{R}_3$. We will solve for c_1 and c_3, and then the ρ_i. Quantities c_1 and c_3 turn out to be functions of the area ratios of sector to triangle y_{12}, y_{23}, and y_{13}.

In Step 4, we carry out a procedure that closely follows Escobal's exposition. We, too, write ρ and ρ for the geocentric distance vector and its magnitude. But our ρ is assumed to point from geocenter to asteroid, while Escobal's ρ points from an Earth-fixed observer to an artificial Earth satellite. Also, our **R** is the sun's geocentric position vector, while Escobal's **R** is a vector pointing from the observer to the geocenter. Finally, our **r** goes from heliocenter to asteroid, while Escobal's **r** goes from geocenter to artificial Earth satellite.

As a preliminary step, we calculate the matrix **a** as the inverse of a matrix, **L**, whose columns are the three observational line-of-sight vectors, $\mathbf{L}^{<i>}$. We do this first, because if **L** is not invertible, i.e., if **L** is singular, we cannot proceed to a solution.

We then compute the modified time intervals τ_1, τ_{13}, and τ_3. We set up the coefficients a, b, and c of the polynomial equation $f(x) = x^8 + ax^6 + bx^3 + c = 0$, and then solve for x, which is actually r_2, the second position vector's magnitude. The r_2 solution leads us to first estimates of ρ_1, ρ_2, ρ_3, $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$.

$$a := L^{-1}$$

$$\|L\| = -0.00010488$$

$$a = \begin{bmatrix} 6.90049508 & -57.77890783 & 227.32973598 \\ -68.45566935 & 243.52716409 & -1021.05583243 \\ 61.19537459 & -185.13402500 & 795.04314884 \end{bmatrix}$$

Note that **a** is the inverse of the matrix **L** having columns $\mathbf{L}^{<1>}$, $\mathbf{L}^{<2>}$, and $\mathbf{L}^{<3>}$.

$$\tau_1 := K \cdot \left(t_1 - t_2\right) \qquad \tau_3 := K \cdot \left(t_3 - t_2\right)$$

$$\tau_{13} := K \cdot \left(t_3 - t_1\right)$$

Compute the modified time differences as functions of the three times $\mathbf{t}_1$, $\mathbf{t}_2$, and $\mathbf{t}_3$.

$$A := \begin{bmatrix} \frac{\tau_3}{\tau_{13}} \\ -1 \\ \frac{-\tau_1}{\tau_{13}} \end{bmatrix} \qquad B := \begin{bmatrix} \left(\left(\tau_{13}\right)^2 - \left(\tau_3\right)^2\right) \cdot \frac{A_1}{6} \\ 0 \\ \left(\left(\tau_{13}\right)^2 - \left(\tau_1\right)^2\right) \cdot \frac{A_3}{6} \end{bmatrix}$$

Calculate the vectors **A** and **B** as per the notation of Escobal.

$$X := \left(R^{\mathrm{T}}\right)^{\langle 1 \rangle} \qquad Y := \left(R^{\mathrm{T}}\right)^{\langle 2 \rangle} \qquad Z := \left(R^{\mathrm{T}}\right)^{\langle 3 \rangle}$$

$$A_2 := -\left(a_{2,1} \cdot A \cdot X + a_{2,2} \cdot A \cdot Y + a_{2,3} \cdot A \cdot Z\right)$$

$$B_2 := -\left(a_{2,1} \cdot B \cdot X + a_{2,2} \cdot B \cdot Y + a_{2,3} \cdot B \cdot Z\right)$$

$$C_\psi := -2 \cdot \left(X_2 \cdot L_{1,2} + Y_2 \cdot L_{2,2} + Z_2 \cdot L_{3,2}\right)$$

$$R_2 := \left(X_2\right)^2 + \left(Y_2\right)^2 + \left(Z_2\right)^2$$

Calculate A^*_2, B^*_2, C_ψ, and R_2^2, as per the notation of Escobal. We have dropped the asterisks on A^*_2 and B^*_2 because such a notation is not permitted in a Mathcad math region: if one types an asterisk, Mathcad interprets this as the multiplication symbol. For a similar reason, we omit the "squared" superscript on R_2^2.

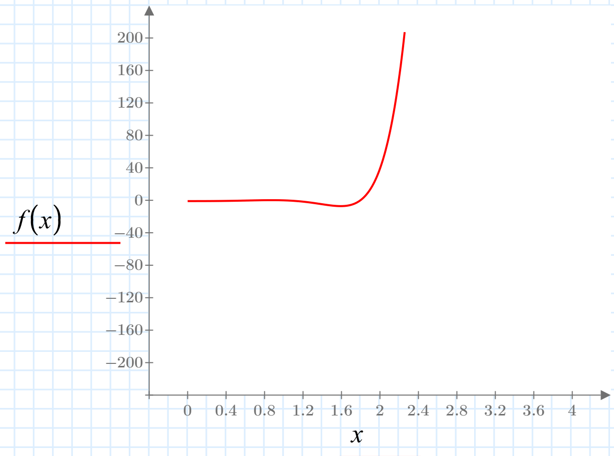
$$aa := -\left(C_\psi \cdot A_2 + A_2^{\,2} + R_2\right) \qquad aa = -3.84651722$$

$$b := -\mu \cdot \left(C_\psi \cdot B_2 + 2 \cdot A_2 \cdot B_2\right) \qquad b = 3.75955423$$

$$c := -\mu^2 \cdot B_2^{\,2} \qquad c = -0.97333874$$

Given aa, b, and c, we solve $f(x) = x^8 + aax^6 + bx^3 + c = 0$ in order to find r_2.

$$f(x) := x^8 + aa \cdot x^6 + b \cdot x^3 + c$$



Mathcad makes it easy to plot f(x) as a function of x, so that we can determine a rough estimate of the root r_2 by inspection. We then use Mathcad's "root" function to refine the rough estimate.

$f(1.79636227) = -0.00000026$

$x := 2$

$r_2 := \mathbf{root}(f(x), x)$

$r_2 = 1.79636227$

$u_2 := \dfrac{\mu}{{r_2}^3}$

$D_1 := A_1 + B_1 \cdot u_2$ $\qquad D_1 = 0.19505016$

$D_3 := A_3 + B_3 \cdot u_2$ $\qquad D_3 = 0.80587207$

$A_1 := \left(a_{1,1} \cdot A \cdot X + a_{1,2} \cdot A \cdot Y + a_{1,3} \cdot A \cdot Z\right)$

$B_1 := \left(a_{1,1} \cdot B \cdot X + a_{1,2} \cdot B \cdot Y + a_{1,3} \cdot B \cdot Z\right)$

$A_3 := \left(a_{3,1} \cdot A \cdot X + a_{3,2} \cdot A \cdot Y + a_{3,3} \cdot A \cdot Z\right)$

$B_3 := \left(a_{3,1} \cdot B \cdot X + a_{3,2} \cdot B \cdot Y + a_{3,3} \cdot B \cdot Z\right)$

Calculate A^*_1, B^*_1, A^*_3, and B^*_3, as per the notation of Escobal. Again, we have dropped the asterisks, as they are not permitted in Mathcad's math region notation.

Upon calculating these four quantities, we are at last able to write down first estimates of ρ_1, ρ_2, and ρ_3, and then first estimates of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$.

$$\rho_1 := \frac{A_1 + B_1 \cdot u_2}{D_1} \qquad r^{\langle 1 \rangle} := \rho_1 \cdot L^{\langle 1 \rangle} - R^{\langle 1 \rangle}$$

$$\rho_2 := A_2 + B_2 \cdot u_2 \qquad r^{\langle 2 \rangle} := \rho_2 \cdot L^{\langle 2 \rangle} - R^{\langle 2 \rangle} \qquad \sqrt{r^{\langle 2 \rangle} \cdot r^{\langle 2 \rangle}} = 1.79636227$$

$$\rho_3 := \frac{A_3 + B_3 \cdot u_2}{D_3} \qquad r^{\langle 3 \rangle} := \rho_3 \cdot L^{\langle 3 \rangle} - R^{\langle 3 \rangle}$$

$$\rho = \begin{bmatrix} 0.89269989 \\ 0.86802982 \\ 0.86699083 \end{bmatrix} \qquad r = \begin{bmatrix} -0.16430796 & -0.29628461 & -0.32788867 \\ 1.62522011 & 1.66837296 & 1.67690825 \\ 0.58622462 & 0.59637615 & 0.59815070 \end{bmatrix}$$

As a matter of Mathcad notation, the subscripts on the ρ_i are true, or "left-bracket" subscripts obtained by typing the ρ symbol, then left bracket ("["), then the subscript number. The subscript number now indicates the row number in the column vector. But the subscripts on A_1, B_1, A_3, and B_3 are symbolic, or "period" subscripts, obtained by typing the symbol, say "A", then period ("."), then the subscript, which is now a part of the symbol. "Period" subscripts are thus a means of distinguishing among scalars when they are not necessarily elements of some vector or matrix.

Another matter of notation is my own notation for vectors and matrices: in every worksheet that works with vectors and matrices, I create a math font called "Vectors & Matrices", which is simply a boldface version of the font for scalars. To see this boldface font, click on any vector or matrix, and note that "Vectors & Matrices" appears as the font tag in the formatting toolbar. (If the formatting toolbar is not visible, click on the View menu, then Toolbars, then click on the word "Formatting" to place a checkmark beside it. The formatting toolbar should now appear.)

A final matter of notation is that Mathcad allows one to compose vectors into the columns of a matrix, and then address the columns of the resulting matrix individually using a "bracketed superscript" notation. We have taken advantage of this capability in setting up the matrices **R**, **r**, and **L** with one column for each of the three observations.

It is clear that **R**, **r**, and **L** are three 3x3 matrices composed of 3x1 column vectors. E.g., **R** is composed of $\mathbf{R}^{<1>}$, $\mathbf{R}^{<2>}$, and $\mathbf{R}^{<3>}$. But we have just calculated ρ as a 3x1 vector with components ρ_1, ρ_2, and ρ_3. We had earlier defined **r** = ρ - **R**, so that ρ = **r** + **R**. If we had needed to, we could have implemented this latter equation for ρ as the three observational equations $\rho^{<i>} = \mathbf{r}^{<i>} + \mathbf{R}^{<i>}$, for i = 1, 2, 3. Then ρ_1, ρ_2, and ρ_3, as we have just defined and calculated them above, would have been the vector magnitudes $|\rho^{<1>}|$, $|\rho^{<2>}|$, and $|\rho^{<3>}|$, respectively.

5. Iterate on the ρ_i with sub-iterations on the three possible area ratios y_{12}, y_{23}, and y_{13}.

We need a velocity-calculating function to be called by function **TWOPOE**.

$$VELO(K, \Delta t, rmag_1, r_1, r_2, y, Arg) := \left\| \begin{array}{l} c \leftarrow C(Arg) \\ s3 \leftarrow \dfrac{K \cdot \Delta t \cdot \left(1 - \dfrac{1}{y}\right)}{c_4} \\ s2 \leftarrow s3^{\frac{2}{3}} \\ f \leftarrow 1 - \dfrac{s2 \cdot c_3}{rmag_1} \\ g \leftarrow \dfrac{K \cdot \Delta t}{y} \\ \left(\dfrac{1}{g}\right) \cdot r_2 - \left(\dfrac{f}{g}\right) \cdot r_1 \end{array} \right.$$

We need function EG ("Extended Gauss") to calculate the area ratios of sector to triangle, y.

$$
\begin{aligned}
EG(K, \Delta t, r_1, r_2) := \; & rmag_1 \leftarrow \sqrt{r_1 \cdot r_1} \\
& rmag_2 \leftarrow \sqrt{r_2 \cdot r_2} \\
& cos\Delta v \leftarrow \sqrt{\frac{1 + \dfrac{r_2 \cdot r_1}{rmag_2 \cdot rmag_1}}{2}} \\
& l \leftarrow \frac{rmag_2 + rmag_1}{4 \cdot \sqrt{rmag_2 \cdot rmag_1} \cdot cos\Delta v} - \frac{1}{2} \\
& m \leftarrow \frac{K^2 \cdot (\Delta t)^2}{\left(2 \cdot \sqrt{rmag_2 \cdot rmag_1} \cdot cos\Delta v\right)^3} \\
& y \leftarrow 0 \\
& y_{new} \leftarrow 1 \\
& \text{while } \left|y - y_{new}\right| \ge 10^{-8} \\
& \quad y \leftarrow y_{new} \\
& \quad x \leftarrow \frac{m}{y^2} - l \\
& \quad \text{if } x \ge 0 \\
& \quad\quad z \leftarrow 4 \cdot \operatorname{asin}\left(\sqrt{x}\right) \\
& \quad\quad Arg \leftarrow z^2 \\
& \quad \text{else} \\
& \quad\quad z \leftarrow 4 \cdot \operatorname{asin}\left(\sqrt{-x}\right) \\
& \quad\quad Arg \leftarrow -z^2 \\
& \quad c \leftarrow C(Arg) \\
& \quad d \leftarrow C\left(\frac{Arg}{4}\right) \\
& \quad X \leftarrow \frac{8 \cdot c_4}{\left(d_2\right)^3} \\
& \quad y_{new} \leftarrow 1 + X \cdot (l + x) \\
& y_{new}
\end{aligned}
$$

We need function **TWOPOE** ("**TWO PO**sition Vector Solution, **E**xtended") to compute the velocity at $\mathbf{r}_1$ needed to attain $\mathbf{r}_2$.

$$
\begin{aligned}
TWOPOE(K, \Delta t, r_1, r_2) := \; & rmag_1 \leftarrow \sqrt{r_1 \cdot r_1} \\
& rmag_2 \leftarrow \sqrt{r_2 \cdot r_2} \\
& cos\Delta v \leftarrow \sqrt{\frac{1 + \dfrac{r_2 \cdot r_1}{rmag_2 \cdot rmag_1}}{2}} \\
& l \leftarrow \frac{rmag_2 + rmag_1}{4 \cdot \sqrt{rmag_2 \cdot rmag_1} \cdot cos\Delta v} - \frac{1}{2} \\
& m \leftarrow \frac{K^2 \cdot (\Delta t)^2}{\left(2 \cdot \sqrt{rmag_2 \cdot rmag_1} \cdot cos\Delta v\right)^3} \\
& y \leftarrow 0 \\
& y_{new} \leftarrow 1 \\
& \text{while } |y - y_{new}| \geq 10^{-8} \\
& \quad y \leftarrow y_{new} \\
& \quad x \leftarrow \frac{m}{y^2} - l \\
& \quad \text{if } x \geq 0 \\
& \quad \quad z \leftarrow 4 \cdot \text{asin}\left(\sqrt{x}\right) \\
& \quad \quad Arg \leftarrow z^2 \\
& \quad \text{else} \\
& \quad \quad z \leftarrow 4 \cdot \text{asin}\left(\sqrt{-x}\right) \\
& \quad \quad Arg \leftarrow -z^2 \\
& \quad c \leftarrow C(Arg) \\
& \quad d \leftarrow C\left(\frac{Arg}{4}\right) \\
& \quad X \leftarrow 8 \cdot c_4 \cdot \left(d_2\right)^{-3} \\
& \quad y_{new} \leftarrow 1 + X \cdot (l + x) \\
& v \leftarrow VELO(K, \Delta t, rmag_1, r_1, r_2, y_{new}, Arg) \\
& \text{augment}(r_1, v \cdot K)
\end{aligned}
$$

We define function **ITER** to allow us to perform a single Gaussian iteration on ρ.

$$ITER(\rho, L, R, K) := \begin{array}{|l} \text{for } i \in 1..3 \\ \quad r^{\langle i \rangle} \leftarrow \rho_i \cdot L^{\langle i \rangle} - R^{\langle i \rangle} \\ y_{12} \leftarrow EG\left(K, \left(t_2 - t_1\right), r^{\langle 1 \rangle}, r^{\langle 2 \rangle}\right) \\ y_{13} \leftarrow EG\left(K, \left(t_3 - t_1\right), r^{\langle 1 \rangle}, r^{\langle 3 \rangle}\right) \\ y_{23} \leftarrow EG\left(K, \left(t_3 - t_2\right), r^{\langle 2 \rangle}, r^{\langle 3 \rangle}\right) \\ c_1 \leftarrow \dfrac{y_{13} \cdot \left(t_3 - t_2\right)}{y_{23} \cdot \left(t_3 - t_1\right)} \\ c_3 \leftarrow \dfrac{-y_{13} \cdot \left(t_1 - t_2\right)}{y_{12} \cdot \left(t_3 - t_1\right)} \\ c_2 \leftarrow -1 \\ G \leftarrow c_1 \cdot R^{\langle 1 \rangle} + c_2 \cdot R^{\langle 2 \rangle} + c_3 \cdot R^{\langle 3 \rangle} \\ \rho_1 \leftarrow \dfrac{\left(a_{1,1} \cdot G_1 + a_{1,2} \cdot G_2 + a_{1,3} \cdot G_3\right)}{c_1} \\ \rho_2 \leftarrow -\left(a_{2,1} \cdot G_1 + a_{2,2} \cdot G_2 + a_{2,3} \cdot G_3\right) \\ \rho_3 \leftarrow \dfrac{\left(a_{3,1} \cdot G_1 + a_{3,2} \cdot G_2 + a_{3,3} \cdot G_3\right)}{c_3} \\ \rho \end{array}$$

We now use function **GAUSS** to iterate on ρ, up to 20 times, as needed.

$$GAUSS(\rho, L, R, K) := \begin{array}{|l} Converged \leftarrow 0 \\ \text{for } j \in 1..20 \\ \quad \text{if } Converged = 0 \\ \quad\quad \rho_{new} \leftarrow ITER(\rho, L, R, K) \\ \quad\quad \delta\rho \leftarrow \left|\rho_{new} - \rho\right| \\ \quad\quad \text{if } \delta\rho \le 0.0001 \\ \quad\quad\quad Converged \leftarrow 1 \\ \quad\quad \rho \leftarrow \rho_{new} \\ \text{for } i \in 1..3 \\ \quad r^{\langle i \rangle} \leftarrow \rho_i \cdot L^{\langle i \rangle} - R^{\langle i \rangle} \\ r \end{array}$$

$$r := GAUSS(\rho, L, R, K)$$

6. Upon convergence of all iterations, calculate $\mathbf{r}_2$ and $\mathbf{v}_2$ in HCI equatorial J2000.0 coordinates.

$$PV := TWOPOE\left(K, \left(t_3 - t_2\right), r^{\langle 2 \rangle}, r^{\langle 3 \rangle}\right)$$

$$r := PV^{\langle 1 \rangle} \qquad r = \begin{bmatrix} -0.29362476 \\ 1.66255252 \\ 0.59481607 \end{bmatrix}$$

$$v := PV^{\langle 2 \rangle} \qquad v = \begin{bmatrix} -0.01076435 \\ 0.00298672 \\ 0.00064000 \end{bmatrix}$$

7. Transform $\mathbf{r}_2$ and $\mathbf{v}_2$ from the HCI equatorial J2000.0 reference frame to the HCI ecliptic J2000.0 reference frame.

$$r := EQEC(r) \qquad r = \begin{bmatrix} -0.29362476 \\ 1.76196635 \\ -0.11559234 \end{bmatrix}$$

$$v := EQEC(v) \qquad v = \begin{bmatrix} -0.01076435 \\ 0.00299484 \\ -0.00060086 \end{bmatrix}$$

(Be sure to see the HISTORICAL NOTE, DISCUSSION OF ASSUMPTIONS, and REFERENCES at the end of this worksheet.)

8. Convert the position and velocity vectors at the second observation time to conic elements.

For this will need function **PVCO** to convert position and velocity to conic elements. **PVCO** invokes function SCAL1, which we define now.

$$SCAL1(K,\alpha,q,e,v) := \begin{array}{l} \text{if } \alpha > 0 \\ \quad E \leftarrow v - 2 \cdot \operatorname{atan}\left(\dfrac{e \cdot \sin(v)}{1+\sqrt{1-e^2}+e\cdot\cos(v)}\right) \\ \quad s \leftarrow \dfrac{E}{\sqrt{\alpha}} \\ \text{else} \\ \quad w \leftarrow \dfrac{1}{K} \cdot \sqrt{\dfrac{q}{1+e}} \cdot \tan\left(\dfrac{v}{2}\right) \\ \quad \text{if } \alpha = 0 \\ \quad\quad s \leftarrow 2 \cdot w \\ \quad \text{else} \\ \quad\quad E \leftarrow 2 \cdot \operatorname{atanh}\left(\sqrt{-\alpha} \cdot w\right) \\ \quad\quad s \leftarrow \dfrac{E}{\sqrt{-\alpha}} \\ s \end{array}$$

Finally, now, we define function **PVCO** ("**P**osition and **V**elocity to **CO**nic Elements").

Note that in **PVCO**, as defined in this document, the subscripts of the **P**, **Q**, and **W** vectors range from 1 through 3 rather than from 0 through 2. Also, the subscripts of **c** range from 1 through 4 rather than from 0 through 3.

$$PVCO(K, r, v) := \left\| \begin{array}{l} rmag \leftarrow \sqrt{r \cdot r} \\ h \leftarrow r \times v \\ hmag \leftarrow \sqrt{h \cdot h} \\ W \leftarrow \dfrac{h}{hmag} \\ E \leftarrow \dfrac{v \cdot v}{2} - \dfrac{K^2}{rmag} \\ \alpha \leftarrow -2 \cdot E \\ p \leftarrow hmag^2 \cdot K^{-2} \\ e \leftarrow \sqrt{1.0 - \dfrac{\alpha \cdot p}{K^2}} \\ q \leftarrow \dfrac{p}{1+e} \\ U \leftarrow \dfrac{r}{rmag} \\ V \leftarrow W \times U \\ \upsilon \leftarrow \text{angle}\left(\dfrac{hmag}{K^2} \cdot v \cdot V - 1.0, \dfrac{hmag}{K^2} \cdot v \cdot U\right) \\ P \leftarrow \cos(\upsilon) \cdot U - \sin(\upsilon) \cdot V \\ Q \leftarrow \sin(\upsilon) \cdot U + \cos(\upsilon) \cdot V \\ i \leftarrow \text{acos}\left(W_3\right) \\ \Omega \leftarrow \text{angle}\left(-W_2, W_1\right) \\ \omega \leftarrow \text{angle}\left(Q_3, P_3\right) \\ s \leftarrow SCAL1(K, \alpha, q, e, \upsilon) \\ c \leftarrow C\left(\alpha \cdot s^2\right) \\ \Delta t \leftarrow q \cdot s + K^2 \cdot e \cdot s^3 \cdot c_4 \\ \begin{bmatrix} q \\ e \\ i \cdot DegPerRad \\ \Omega \cdot DegPerRad \\ \omega \cdot DegPerRad \\ \Delta t \end{bmatrix} \end{array} \right.$$

We now invoke **PVCO** and place its output into array **CONIC**.

$$CONIC := PVCO(K, r, v)$$

$$CONIC = \begin{bmatrix} 0.75167393 \\ 0.47817689 \\ 4.05977204 \\ 213.71260957 \\ 103.32076351 \\ 169.94658789 \end{bmatrix}$$

We should note that the position vector input to **PVCO** must have units of A.U. and the velocity vector must have units of A.U. per day. We summarize the uniform Gaussian angles-only orbital solution as follows.

$CONIC_1 = 0.75167393$	Perihelion distance in A.U.
$CONIC_2 = 0.47817689$	Path eccentricity.
$CONIC_3 = 4.05977204$	Path inclination, in degrees.
$CONIC_4 = 213.71260957$	Celestial longitude of ascending node, in degrees.
$CONIC_5 = 103.32076351$	Argument of perihelion, in degrees.
$CONIC_6 = 169.94658789$	Time of flight from perihelion to epoch, in days.
$a := \dfrac{CONIC_1}{1 - CONIC_2}$	Orbital quantities defined only for an elliptical path:
$a = 1.44047651$	Semimajor axis, A.U.

$$n := K \cdot a^{\frac{-3}{2}} \cdot DegPerRad$$

$n = 0.57009181$ Mean motion, deg/day

$$P := \frac{360.0}{365.25 \cdot n}$$

$P = 1.72889043$ Orbital period, Julian years

$$M := n \cdot CONIC_6$$

$M = 96.88515854$ Mean anomaly, degrees

When we subtract the time of flight from perihelion to epoch, $\mathbf{CONIC}_6$, from epoch, $\mathbf{t}_2$, we get the Julian ephemeris date of perihelion passage:

$$t_2 - CONIC_6 = 2450631.25107$$

This Julian ephemeris date works out to 1997 July 1.75107. Here is a summary of our Gaussian preliminary two-body solution, using just three observations from Minor Planet Electronic Circular MPEC 1997-Y11, together with the Minor Planet Center's definitive, perturbed solution using 19 observations over the time span 1997 December 6-21.

Orbital Element/Parameter	Gaussian Value	MPEC 1997-Y11 Value
Time of perihelion passage, TT	1997 Jul 1.75107	1997 Jul 1.37109
Eccentricity	0.4781769	0.4823930
Perihelion distance, A.U.	0.75167393	0.74626491
Argument of perihelion*, deg	103.32076	102.69821
Longitude of Asc. Node*, deg	213.71261	214.03784
Inclination*, deg	4.05977	4.08628
Semimajor axis, A.U.	1.4404765	1.4417597
Mean motion, deg/day	0.57009181	0.56933087
Orbital period, Julian years	1.72889043	1.73120120

*Angles are referred to mean ecliptic and equinox of J2000.0.

HISTORICAL NOTE

The asteroid 1997 XF_{11} made international headlines when it was determined in March 1998 that it might strike Earth on October 26, 2028. For more information about the incident, see the following contemporaneous references.

Gareth V. Williams, Minor Planet Electronic Circular MPEC 1997-Y11 (23 December 1997).

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Adam Rogers and Sharon Begley, "Never Mind!", *Newsweek,* March 23, 1998.

Malcolm W. Browne, "In 2028, the sky may fall," *The Denver Post,* Thursday, March 12, 1998.

Joseph C. Anselmo, "Asteroid Search", *Av. Week & Space Technology,* March 23, 1998, p. 21.

Ron Cowen, "Near-Earth asteroid: A far miss," *Science News,* Vol. 153 (March 21, 1998), p. 185.

(For more information about this "Gauss's Angles-Only Method" worksheet and related Mathcad worksheets, see "Mathcad Worksheets by Astroger" at http://astroger.com.)

DISCUSSION OF OF ASSUMPTIONS

We should take a moment to consider the assumptions made in this worksheet, and how they affect the validity of our solution.

a. We have employed UPM, a method of two-body orbit propagation valid for any path eccentricity. Since the asteroid was not very close to any major planets, and moved through an arc of only about six degrees during the time span of the observations, the assumption of two-body mechanics seems reasonable.

b. We have ignored "light-time", the amount if time it took for light from the asteroid to reach the observing telescopes on Earth. This is a small effect for the problem at hand: asteroid 1997 XF_{11} was about 0.9 A.U. away when the three observations used in this worksheet were actually made. So the light-time correction amounts to less than eight minutes.

c. We have treated the topocentric, angles-only observations as though they were geocentric, i.e., we have ignored the fact that the observers were on Earth's surface, and not at the geocenter. Since Earth's radius is about 1/23454.8 A.U., it seems we have ignored a rather small effect.

d. Function **SUNPOS** is a "low-precision", analytical model of the sun's apparent motion in the geocentric equatorial J2000.0 reference frame. More accurate solar ephemeris models are available, e.g., those in VSOP87 and the JPL ephemerides on CD-ROM. But adopting either model would further complicate an already rather complicated worksheet.

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[9] J. M. A. Danby, *op. cit.,* Section 6.9.

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