ON CALCULATING THE PHOTOPERIOD
IN PLANT PHYSIOLOGY

Roger L. Mansfield
Astronomical Data Service
http://astroger.com
January 12, 2004

(Updated to PTC's Mathcad Prime 10.0 on 2024 July 25)

Many plant species exhibit physiological responses to the duration of daylight, or to changes in the duration of daylight, as the days progress through the tropical year. "Photoperiodism" is the name that plant physiologists have given to this phenomenon, while "photoperiod" is the name given to the duration of daylight. I first became aware of these concepts as the result of some calculations that I did for Dr. Frank B. Salisbury, in support of his textbook in plant physiology [1].

The Photoperiod Equation. Recently, at the request of Brazilian plant physiologist Dr. Lilian B. P. Zaidan, I developed a Mathcad worksheet that calculates the photoperiod as a function of day number and geographic latitude [2]. In that worksheet, I calculated the photoperiod by subtracting the time of sunrise from the time of sunset for each day of the year 2003, for varying geographic latitudes (from 5°S to 25°S in 5° increments, for Brazil).

I then showed, in an addendum to the worksheet, that one can deduce a simpler formula from the more rigorous astronomical calculations. The simpler formula agrees with the more rigorous astronomical calculations to within about 2-3 minutes.

But upon further reflection, I have come to realize that the simpler formula can be made even more simple, and yet more accurate, with a just few changes. The purpose of this worksheet, then, is to show how to make the changes needed to arrive at an even simpler, yet more accurate formula.

We will need the following constants from the cited worksheet in what follows.

$$DegPerRad := \frac{180}{\pi}$$ Number of degrees in one radian.

$$\omega_E := \frac{360.98564735}{DegPerRad}$$ Earth's rotation rate in radians/day.

$$R := \frac{50}{60} \cdot \frac{1}{DegPerRad}$$ Twilight factor (set to its value for sunrise and sunset).

Given these constants, we now can now restate the "Photoperiod Equation" and the two examples of its usage that were provided in the addendum to [2]. For the Sun's declination, δ, and the geographic latitude, L, we have, in minutes,

$$Photoperiod(\delta, L) := \frac{\pi + 2 \cdot \operatorname{asin}\left(\frac{\sin(R) + \sin(L) \cdot \sin(\delta)}{\cos(L) \cdot \cos(\delta)}\right)}{\omega_E} \cdot 1440$$

At the southern hemisphere winter and summer solstices in the year 2003, for which δ = 23.4389 degrees and δ = -23.4389 degrees, respectively, we obtain for latitude 25°S:

$$Photoperiod\left(\frac{23.4389}{DegPerRad}, \frac{-25}{DegPerRad}\right) = 633.1$$

$$Photoperiod\left(\frac{-23.4389}{DegPerRad}, \frac{-25}{DegPerRad}\right) = 819.3$$

The values computed by rigorous astronomical calculation are 635.0 and 821.8 minutes, respectively. This is good agreement, but is it possible to improve the accuracy of the photoperiod equation just given, without further complicating it? I realized today that the answer is yes.

I came to this realization as follows. First, it troubled me that the agreement was not better, so I asked myself how a 1.9 minute error at the southern winter solstice and a 2.5 minute error at the summer solstice could arise. I then considered that ω_E, Earth's rotation rate, is measured with respect to the moving vernal equinox, and not with respect to the moving Sun.

The fact that the Sun moves eastward at a mean rate of 0.98564735 degrees per day, in the same direction that Earth rotates, means that the actual photoperiod is longer (by about two minutes at the spring and fall equinoxes) than calculated by the photoperiod equation given above. We want to modify this photoperiod equation so that we can account for this effect all year long. Thus we see that we do not want ω_E in the denominator, but rather we want

ω_E^* = (360.98564735 - 0.98564735)/DegPerRad = 360/DegPerRad = 2π radians.

If we divide the numerator on the right side of the photoperiod equation by ω_E^* instead of by ω_E, we obtain the following equation.

$$Photoperiod(\delta, L) := \frac{\pi + 2 \cdot \operatorname{asin}\left(\frac{\sin(R) + \sin(L) \cdot \sin(\delta)}{\cos(L) \cdot \cos(\delta)}\right)}{2 \cdot \pi} \cdot 1440$$

Upon further simplification, we have the following.

Simplified Photoperiod Equation

$$Photoperiod(\delta, L) := 720 + \frac{1440}{\pi} \cdot \operatorname{asin}\left(\frac{\sin(R) + \sin(L) \cdot \sin(\delta)}{\cos(L) \cdot \cos(\delta)}\right)$$

With this simplified photoperiod equation, we obtain southern hemisphere summer and winter solstice values as follows.

$$Photoperiod\left(\frac{23.4389}{DegPerRad}, \frac{-25}{DegPerRad}\right) = 634.9$$

$$Photoperiod\left(\frac{-23.4389}{DegPerRad}, \frac{-25}{DegPerRad}\right) = 821.5$$

The summer solstice value is off by only about 0.1 minute, and the winter solstice value is off by only about 0.3 minute.

Our final Simplified Photoperiod Equation is much more descriptive as well. It says that the photoperiod is half a day, plus or minus a term that depends upon δ, L, and R. This description makes sense, and also makes the photoperiod equation much easier to remember.

The Photoperiod Rate Equation. As noted above, both the photoperiod and its diurnal rate of change are of interest in plant physiology. We can easily calculate the diurnal rate by subtracting one day's photoperiod from the next, and then dividing by one day (i.e., by unity). Then we have the photoperiod rate in minutes per day.

However, just as we were able to find an analytical expression for the photoperiod, we can find an analytical expression for the photoperiod rate, by differentiating the photoperiod equation with respect to time. We do this by first differentiating the Sun's declination δ with respect to day number d. Then we differentiate the photoperiod with respect to δ. By the "chain rule" of the calculus, the diurnal rate of change of photoperiod is then the product (D_δ Photoperiod) x (D_d δ).

To see how to differentiate the Sun's declination with respect to d, let us revisit our equations for calculating the declination, δ, as given in [2]. We have for the year 2003 that

$$e_o := 0.01670785 \qquad i_o := \frac{23.4389}{DegPerRad}$$

$$\omega_o := \frac{282.9920}{DegPerRad} \qquad M_o := \frac{356.2657}{DegPerRad}$$

$$A_1 := 2 \cdot e_o \qquad A_2 := \frac{5}{4} \cdot {e_o}^2$$

$$B_1 := \cos(i_o) \qquad B_2 := \sin(i_o)$$

Now, writing out the equations needed to calculate the Sun's declination, we have

$$M(d) := M_o + \frac{0.9856 \cdot d}{DegPerRad}$$

$$\upsilon(M) := M + A_1 \cdot \sin(M) + A_2 \cdot \sin(2 \cdot M)$$

$$u(\upsilon) := \text{mod}(\upsilon + \omega_o, 2 \cdot \pi)$$

$$\delta(u) := \text{asin}(B_2 \cdot \sin(u))$$

The corresponding derivative functions are as follows.

$$M_d(d) := \frac{0.9856}{DegPerRad}$$

$$\upsilon_M(M) := 1 + A_1 \cdot \cos(M) + 2 \cdot A_2 \cdot \cos(2 \cdot M)$$

$$u_\upsilon(\upsilon) := 1$$

$$\delta_u(u) := \frac{B_2 \cdot \cos(u)}{\sqrt{1 - (B_2 \cdot \sin(u))^2}}$$

By the chain rule, we can obtain the derivative of δ with respect to d by multiplying the successive derivatives.

$$\delta_d(d) := \delta_u(u(\upsilon(M(d)))) \cdot 1 \cdot \upsilon_M(M(d)) \cdot M_d(d)$$

To differentiate photoperiod with respect to δ, we let Mathcad assist via its Maple symbolic differentiation capability. Using x temporarily in place of δ, we obtain the following.

$$\frac{d}{dx} Photoperiod(x, L) \rightarrow \frac{1440 \cdot \sin(x) \cdot \sin\left(\frac{\pi}{216}\right) + 1440 \cdot \sin(L) \cdot \sin(x)^2 + 1440 \cdot \cos(x)^2 \cdot \sin(L)}{\pi \cdot \cos(L) \cdot \cos(x)^2 \cdot \sqrt{-\frac{\sin\left(\frac{\pi}{216}\right)^2 + 2 \cdot \sin(L) \cdot \sin(x) \cdot \sin\left(\frac{\pi}{216}\right) + \sin(L)^2 \cdot \sin(x)^2}{\cos(L)^2 \cdot \cos(x)^2} + 1}}$$

This simplifies to the rate of change of photoperiod, with respect to declination, as follows.

$$Photorate(\delta, L) := \frac{1440}{\pi} \cdot \frac{\tan(L) + \dfrac{\sin(R) + \sin(L) \cdot \sin(\delta)}{\cos(L) \cdot \cos(\delta)} \cdot \tan(\delta)}{\left(1 - \dfrac{(\sin(R) + \sin(L) \cdot \sin(\delta))^2}{\cos(L)^2 \cdot \cos(\delta)^2}\right)^{\frac{1}{2}}}$$

Now the photoperiod rate that we seek is a product of this derivative and $\delta_d(d)$.

$$\delta 1(d) := \delta(u(v(M(d))))$$

$$Photorate1(d, L) := Photorate(\delta 1(d), L) \cdot \delta_d(d)$$

It is easy to check this formula. Let's do it for, say, day number 49 (February 18).

$$Photorate1\left(49, \frac{-25}{DegPerRad}\right) = -1.4$$

$$Photoperiod\left(\delta 1(49), \frac{-25}{DegPerRad}\right) = 772.5$$

$$Photoperiod\left(\delta 1(50), \frac{-25}{DegPerRad}\right) = 771.1$$

$$Photoperiod\left(\delta 1(50), \frac{-25}{DegPerRad}\right) - Photoperiod\left(\delta 1(49), \frac{-25}{DegPerRad}\right) = -1.4$$

Now let's do a plot for the entire year 2003, for latitudes 25°N, 50°N, and 65°N, comparing the analytically computed photoperiod rates with the numerically computed photoperiod rates.

```
Curve1(N, L) := || for i ∈ 1 .. N
                 ||    || Rate_i ← Photoperiod(δ1(i+1), L) − Photoperiod(δ1(i), L)
                 || Rate
```

$$Curve2(N, L) := \left\| \begin{array}{l} \text{for } i \in 1..N \\ \quad \| \; Rate_i \leftarrow Photorate1(i, L) \\ Rate \end{array} \right.$$

$$Curve25N1 := Curve1\left(366, \frac{25}{DegPerRad}\right)$$

$$Curve25N2 := Curve2\left(366, \frac{25}{DegPerRad}\right)$$

$$Curve50N1 := Curve1\left(366, \frac{50}{DegPerRad}\right)$$

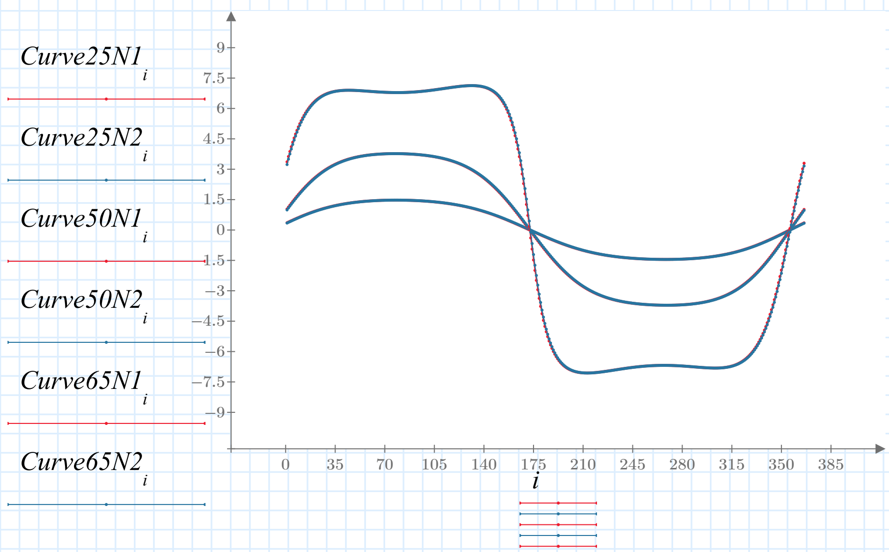
$$Curve50N2 := Curve2\left(366, \frac{50}{DegPerRad}\right)$$

$$Curve65N1 := Curve1\left(366, \frac{65}{DegPerRad}\right)$$

$$Curve65N2 := Curve2\left(366, \frac{65}{DegPerRad}\right)$$

Need to tell Mathcad Prime the range of values for the range variable i in the plot:

$$i := 1, 2..366$$



We see that the blue dots are superimposed so closely over the red dots that we can no longer see the red dots in most instances. Clearly, our photoperiod rate equation is correct.

Concluding Remarks. Analytical expressions for both the photoperiod and its diurnal rate, as functions of day number and geographic latitude, have been obtained.

It should also be noted that:

a. The photoperiod rate, in minutes per day, as obtained by subtracting one day's photoperiod from the next day's, is very close to the analytically computed photoperiod rate.

b. The analytically computed photoperiod rate has been demonstrated to be correct, but it is a rather complicated function of sines, cosines, tangents, and a square root.

Thus we are justified in filing away the photoperiod rate equation as an interesting mathematical curiosity, and in adopting the Simplified Photoperiod Equation as our preferred tool for studies of photoperiodism.

REFERENCES

[1] Salisbury, Frank B. and Ross, Cleon W., Plant Physiology, Third Edition (1985), Wadsworth Publishing Company, Belmont, California. See Michael J. Salisbury's photoperiod and photoperiod rate graphs on p. 426; see also pp. 427 and 438.

[2] Mansfield, Roger L., "Plant Photoperiod vs. Day Number, as a Function of Geographic Latitude," Mathcad 11 Worksheet, Astronomical Data Service, 3922 Leisure Lane, Colorado Springs, CO 80917-3502 U.S.A., December 19, 2003.