## BATCH LEAST SQUARES DIFFERENTIAL CORRECTION OF A GEOCENTRIC ORBIT

PART 2 - MANUAL CORRECTION WORKSHEET

Roger L. Mansfield, June 1, 2025 http://astroger.com/

In this worksheet we differentially correct (DC) the orbit of an artificial Earth satellite or space probe using a test case specified in worksheet Gd1, or in a worksheet derived from Gd1. You should open worksheet Gd1, or your own worksheet derived from Gd1, and click on "Calculate Worksheet" from the Math menu now, if you have not already done so.

The process that we will use in this worksheet is documented in Refs. [1] and [2] for the differential correction of Earth orbits using radar observations. However, we will use optical observations in this worksheet. The batch equation of differential correction (BEDC) is:

$$X_{o}' = X_{o} + (A^{T}WA)^{-1} A^{T}W [Y - F(X_{o})].$$

Here  $X_o$  is the initial estimate of the state vector, i.e., position and velocity, at epoch  $t_o$ .  $X_o'$  is the "improved" estimate of  $X_o$  at  $t_o$ , obtained by adding  $(A^TWA)^{-1} A^TW [Y - F(X_o)]$  to  $X_o$ .

If we let n be the number of observations, then Y is a 2n-by-1 column vector of measurements, since for our problem in geocentric motion the measurements are topocentric right ascension (RA, or  $\alpha$ ) and topocentric declination (DEC, or  $\delta$ ). If we denote the total number of measurements by N, then N = 2n.

 $F(X_o)$  is thus an N-by-1column vector of "computed" measurements. What this means is that the RA and DEC for each observation are computed via our UPM model of two-body motion, by propagating the current estimate,  $X_o$  to the observation times  $t_i$  for i = 1, ..., n, and by then computing the topocentric RA and DEC at each observation time, given the specified location of the observer. We say "current estimate,  $X_o$ " because we will find it necessary to iterate on the BEDC, testing for convergence at each iteration by means of a criterion we will define below. If we have convergence on a given iteration, then we stop and convert the solution to conic elements. But if we do not have convergence, then we replace  $X_o$  by  $X_o$ ' and solve the BEDC again, i.e., iterate. (We could also implement an iteration counter and stop the DC if some maximum allowable number of iterations is reached without convergence, but that is not needed here because we iterate the BEDC manually by clicking on "Calculate Worksheet".)

[Y - F(X<sub>o</sub>)] is the N-by-1 column vector of residuals, in the sense "observed minus computed". The BEDC is a form of the least squares normal equations, N equations in six unknowns, which result when one answers the question, "what is a necessary condition that the weighted sum of squares of the residuals be a minimum?" The residuals are not actually  $\Delta \alpha$  and  $\Delta \delta$ , but rather  $\cos \delta \Delta \alpha$  and  $\Delta \delta$ ; they are the projections of  $\Delta L$  on **A** and **D** in turn. (The  $\cos \delta$  factor can become quite important when the object passes near a celestial pole, where large changes in  $\alpha$  accompany relatively small changes in arc length in the direction of motion.)

A, the "A-matrix", is the N-by-6 array of partial derivatives of the N measurements with respect to the six components of the state vector  $X_o$ . We will compute the A-matrix from the O-matrix and the G-matrix, i.e., A = OG. O is the N-by-6 matrix of partials of the measurements with respect to the state vector at observation times  $t_i$ , for i = 1, ..., n. G is Goodyear's 6-by-6 state transition matrix, i.e., the 6-by-6 matrix of partials of the state components at times  $t_i$  with respect to the state components at  $t_o$ . G is therefore a 6-by-6 Jacobian matrix defined at each observation time  $t_i$  for i = 1, ..., n.

W is the weight matrix. Under the assumption that the measurements are Gaussian random variables, and are not correlated (Danby [3] has a good discussion of this), W is a diagonal matrix and each diagonal entry is  $1/\sigma_i^2$ , where  $\sigma_i^2$  is the variance of measurement i. (In Part 1, we specified the N-by-N identity matrix with values of  $\sigma_i = 1.0$  radians. More realistic sigmas for the RA and DEC measurements, e.g., the number of radians in one arc-second, would improve the statistics, and yet not change the solution state vector.)

Here now is an outline of the steps we will follow:

1. Retrieve the test case values from disk, as specified by worksheet Gd1, or as specified by your own worksheet that was derived from Gd1 by duplication and modification.

Retrieval includes obtaining the initial or current estimate of state, X, and the RMS history matrix. Each time you click on "Calculate Worksheet," GDC performs another iteration of weighted, batch least squares differential correction. At each iteration the corrected values of X are written to disk along with the RMS for that iteration. The corrected values of X thus become the current state estimate for the next iteration, and the RMS history is accumulated so that you can keep track of how the DC is going.

2. Define the functions needed in the DC: C, FG, GMAT, and FXA.

3. Obtain the computed measurements, FX, and the A-matrix, A, by invoking FXA.

4. Compute the residuals,  $\Delta Y$ , the A<sup>T</sup>WA matrix ATWA, and the A<sup>T</sup>W $\Delta Y$  matrix, ATW $\Delta Y$ .

5. Solve for and apply the corrections to state,  $\Delta X$ . Compute the current RMS, display the RMS history, and test for convergence.

6. Write the corrected state vector to disk and convert to conic elements.

7. Repeat Steps 1-6, by clicking on "Calculate Worksheet", until convergence is obtained.

As a preliminary, we define some constants that we will need, and set the Mathcad worksheet ORIGIN to 1 so that subscripts start at unity rather than at zero.

ת	egPerRad	180
	egiernau	π

**ORIGIN**  $\equiv 1$ 

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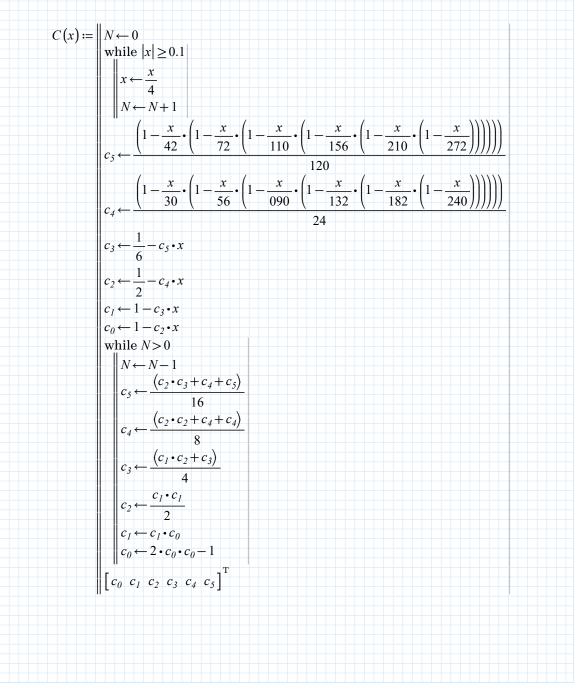
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SecPerDeg := 3600.0	Earth's mean equatorial radius in km:
$SecPerRev := SecPerDeg \cdot 360.0$	$a_e := 6378.137$
Retrieve the test case values from disk, as s our own worksheet that was derived from Go	specified by worksheet Gd1, or as specified by d1 by duplication and/or modification.
$n \coloneqq \text{READPRN} (\text{"NOBS.prn"})_1$	Number of observations.
$t \coloneqq \text{READPRN}(\text{"TVALS.prn"})$	Observation times.
W := READPRN ("WEIGHTS.prn")	Measurement weights matrix.
<i>R</i> := READPRN ("RVALS.prn")	Values of <b>R</b> .
Y:= READPRN ("YVALS.prn")	Values of Y.
X := READPRN ("STATE.prn")	State vector (corrected by Gdc).
<i>RMS</i> := READPRN ("RMS.prn")	RMS history for state corrections by Gdc (one entry for each iteration).
$N := 2 \cdot n$	Set number of measurements.
<i>k</i> := 0.07436684771154	Set WGS-84 Gaussian constant for geocentric orbital motion. See [10].
μ:=1	Assume that mass of secondary (artificial Earth satellite or space probe) is negligible relative to mass of primary
$K := k \cdot \sqrt{\mu}$	(Earth).
<i>n</i> = 8	Display number of observations retrieved from disk.

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2. Define the functions needed in the DC: C, FG, GMAT, and FXA.

For path propagation one needs to calculate only  $c_0$  through  $c_3$ , but for the state transition matrix, G, one needs  $c_0$  through  $c_5$ . To keep down the length of this worksheet we define one version of **C**, the one that calculates  $c_0$  through  $c_5$ . (Remember that since the ORIGIN = 1, the subscripts of the c-functions that we will use outside of the function **C** will range from 1 through 6, rather than from 0 through 5.)



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Function UKEP solves the uniform Kepler equation for function FG. FG, in turn, propagates position and velocity for function FXA.  $\begin{aligned} UKEP\left(\tau, rmag_{o}, \sigma_{o}, \alpha\right) &\coloneqq \left| \begin{array}{c} s \leftarrow \frac{\tau}{rmag_{o}} \\ \Delta s \leftarrow s \\ \text{while } |\Delta s| \geq 0.00000001 \\ \left| \begin{array}{c} c \leftarrow C\left(\alpha \cdot s^{2}\right) \\ F \leftarrow rmag_{o} \cdot s \cdot c_{2} + \sigma_{o} \cdot s^{2} \cdot c_{3} + s^{3} \cdot c_{4} - \tau \\ DF \leftarrow rmag_{o} \cdot c_{1} + \sigma_{o} \cdot s \cdot c_{2} + s^{2} \cdot c_{3} \\ DDF \leftarrow \sigma_{o} \cdot c_{1} + (1 - rmag_{o} \cdot \alpha) \cdot s \cdot c_{2} \\ \text{if } DF > 0 \end{aligned} \end{aligned}$  $\| \begin{array}{c} \text{if } DF \ge 0 \\ \| m \leftarrow 1 \\ \text{else} \\ \| m \leftarrow -1 \\ \Delta s \leftarrow \frac{-5 \cdot F}{\left( DF + m \cdot \sqrt{\left| (4 \cdot DF)^2 - 20 \cdot F \cdot DDF \right|} \right)} \\ s \leftarrow s + \Delta s \end{array}$  $FG(K, r_o, v_o, \Delta t) := \begin{vmatrix} \tau \leftarrow K \cdot \Delta t \\ rmag_o \leftarrow \sqrt{r_o \cdot r_o} \\ \sigma_o \leftarrow r_o \cdot v_o \\ a \leftarrow \frac{2}{rmag_o} - v_o \cdot v_o \\ s \leftarrow UKEP(\tau, rmag_o, \sigma_o, a) \\ c \leftarrow C(a \cdot s^2) \\ f_r \leftarrow 1 - s^2 \cdot c_3 \cdot rmag_o^{-1} \\ c \leftarrow t - s^3 - c \end{vmatrix}$  $g_r \leftarrow \tau - s^3 \cdot c_4$  $rmag \leftarrow rmag_{o} \cdot c_{1} + \sigma_{o} \cdot s \cdot c_{2} + s^{2} \cdot c_{3}$  $\begin{cases} f_{v} \leftarrow -s \cdot c_{2} \cdot (rmag \cdot rmag_{o})^{-1} \\ g_{v} \leftarrow 1 - s^{2} \cdot c_{3} \cdot rmag^{-1} \\ \begin{bmatrix} K \ \alpha \ rmag_{o} \ f_{r} \ f_{v} \\ \tau \ s \ rmag \ g_{r} \ g_{v} \end{bmatrix}$ 

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Function **GMAT** provides the state transition matrix for function **FXA**.

The state transition matrix formulation that we use below is based upon the seminal works of Goodyear [4], [5]. See also Shepperd [6], Battin [7], and Der [8] for more recent expositions.

Before defining **GMAT**, we define functions  $S_{11}$ ,  $S_{12}$ ,  $S_{21}$ , and  $S_{22}$  just to make **GMAT** fit horizontally and vertically within the margins of a single Mathcad page.

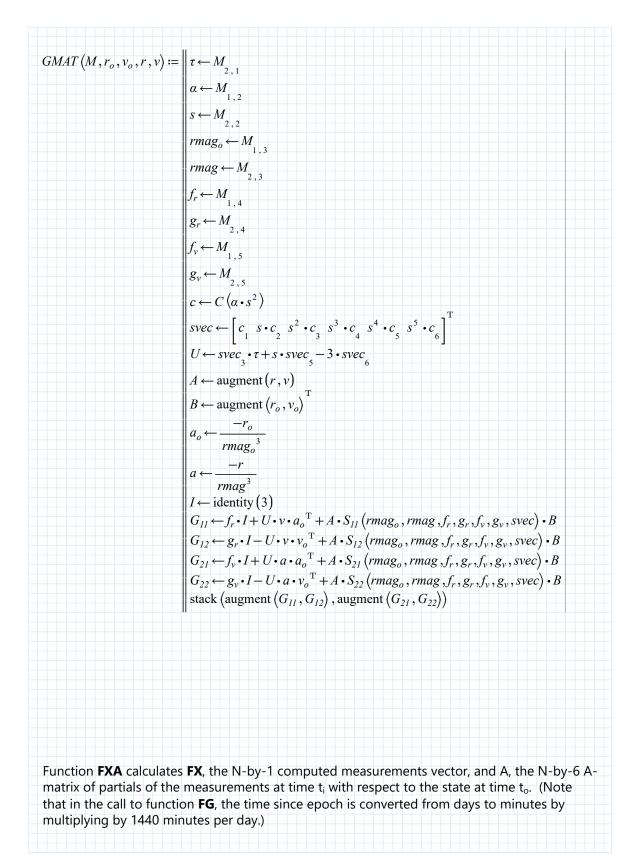
$$S_{II}(rmag_{o}, rmag, f_{r}, g_{r}, f_{v}, g_{v}, s) := \begin{bmatrix} f_{v} \cdot s_{2} + \frac{f_{r} - 1}{rmag_{o}} & \\ -\frac{f_{v} \cdot s_{3}}{rmag_{o}} & -f_{v} \cdot s_{3} \\ \frac{(f_{r} - 1) \cdot s_{2}}{rmag_{o}} & (f_{r} - 1) \cdot s_{3} \end{bmatrix}$$

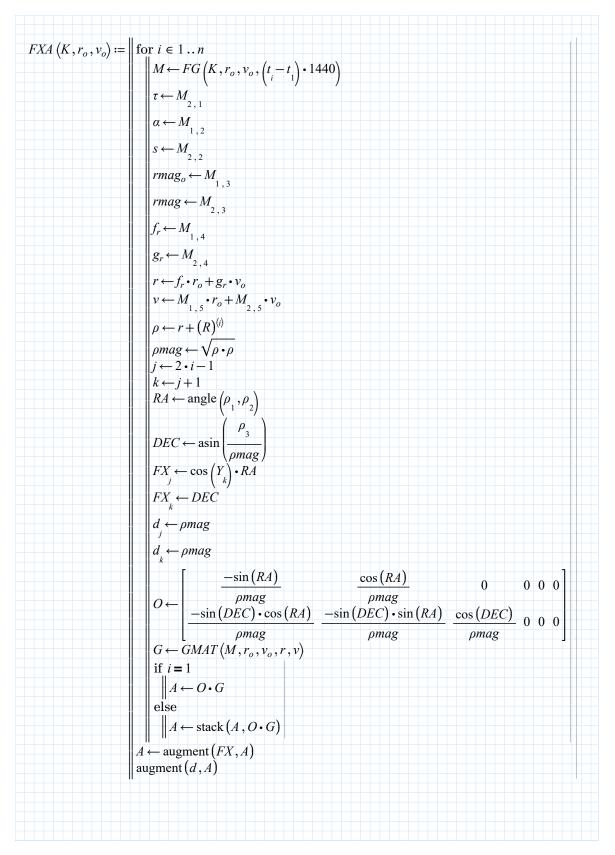
$$S_{12}(rmag_{o}, rmag, f_{r}, g_{r}, f_{v}, g_{v}, s) \coloneqq \begin{bmatrix} -f_{v} \cdot s_{3} & -(g_{v} - 1) \cdot s_{3} \\ (f_{r} - 1) \cdot s_{3} & g_{r} \cdot s_{3} \end{bmatrix}$$

$$S_{21}(rmag_{o}, rmag, f_{r}, g_{r}, f_{v}, g_{v}, s) := \begin{bmatrix} -f_{v} \cdot \left(\frac{s_{1}}{rmag_{o} \cdot rmag} + \frac{1}{rmag^{2}} + \frac{1}{rmag_{o}^{2}}\right) - \frac{f_{v} \cdot s_{2} + \frac{g_{v} - 1}{rmag}}{rmag} \\ \frac{f_{v} \cdot s_{2} + \frac{(f_{r} - 1)}{rmag_{o}}}{rmag_{o}} - \frac{f_{v} \cdot s_{2} + \frac{g_{v} - 1}{rmag}}{f_{v} \cdot s_{3}} \end{bmatrix}$$

$$S_{22}(rmag_{o}, rmag, f_{r}, g_{r}, f_{v}, g_{v}, s) \coloneqq \begin{bmatrix} -\frac{f_{v} \cdot s_{2} + \frac{g_{v} - 1}{rmag}}{rmag} & -(g_{v} - 1) \cdot s_{2} \\ -\frac{g_{v} - 1}{rmag} & -(g_{v} - 1) \cdot s_{2} \\ -\frac{g_{v} - 1}{rmag} & -(g_{v} - 1) \cdot s_{2} \end{bmatrix}$$

(Note that because ORIGIN = 1, the subscripts of the c-functions and Goodyear's s-functions range from 1 to 6 rather than from 0 to 5. It is especially important to note this difference when checking the **GMAT** formulas against Goodyear's original works.)

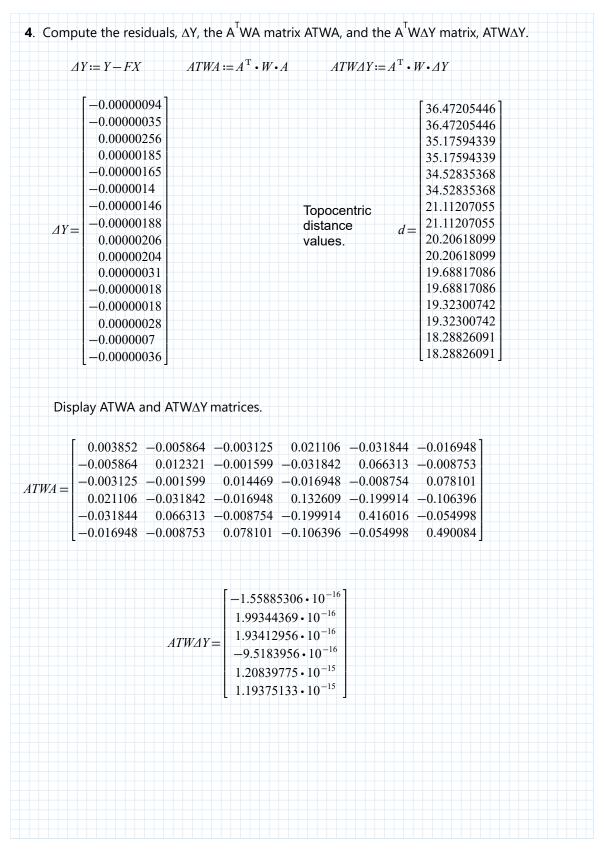




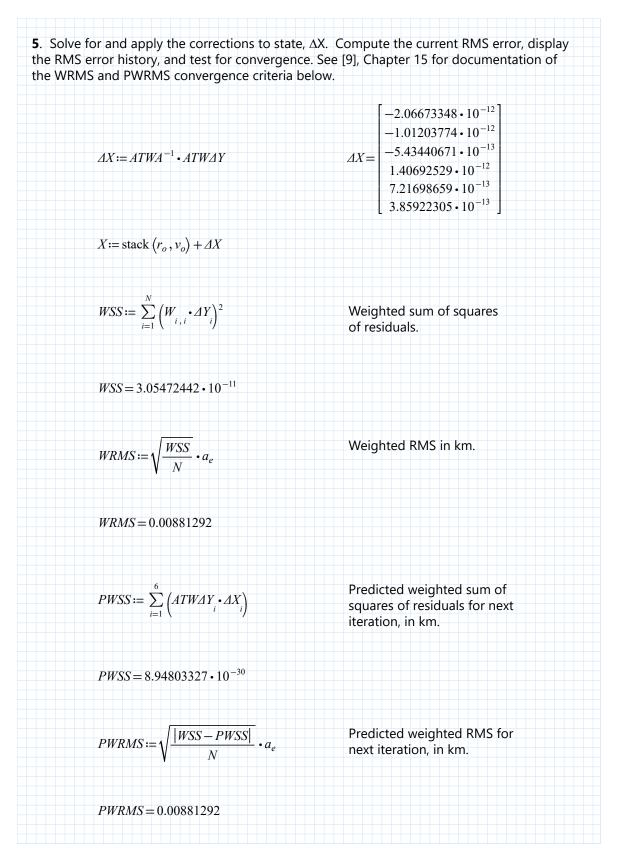
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	$r_o \coloneqq \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$	$v_o \coloneqq \begin{bmatrix} X_4 \\ X_5 \\ X_6 \end{bmatrix} \cdot \frac{1}{K}$	
		$\begin{bmatrix} 5\\ X_6 \end{bmatrix}$ K	
	$M \coloneqq FXA\left(K, r_o\right)$	$(v_o, v_o)$	
	$FX := M^{(2)}$	A := submatrix (M, 1, N, 3, 8) Extract topocentric distance values for information about computed ranges for each of the observations.	
		$d := M^{(1)}$	
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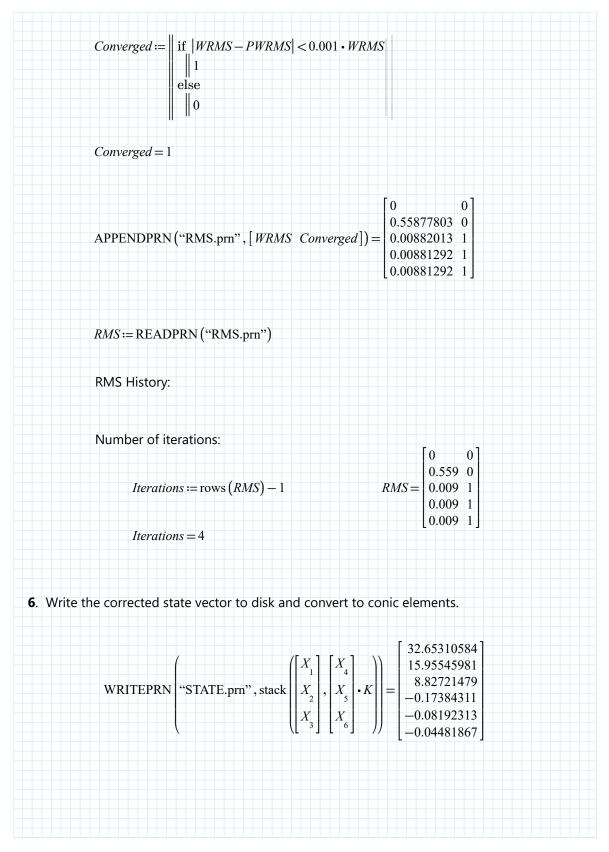
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Compute and display the conic elements by calling function **PVCO** to transform position and velocity to conic elements.

$$r_{I} \coloneqq \begin{bmatrix} X_{1} \\ X_{2} \\ X_{3} \end{bmatrix} \qquad v_{I} \coloneqq \begin{bmatrix} X_{4} \\ X_{5} \\ X_{6} \end{bmatrix} \cdot K \qquad \frac{v_{I}}{K} = \begin{bmatrix} -2.33764262 \\ -1.10160821 \\ -0.60267008 \end{bmatrix}$$

**PVCO** needs position in<br/>E.R. But here is position<br/>in units of km:**PVCO** needs velocity in<br/>E.R./min. But here is<br/>velocity in units of km/sec:

[208265.982516]	
$r_1 \cdot a_e = 101766.108587$	$v_1 \cdot \frac{u_e}{10} = -8.7086158$
56301.18527	60 -4.76432736

 $rmagl := \sqrt{r_1 \cdot r_1}$  rmagl = 37.3994885 E.R.

PVCO also invokes function SCAL1, which we define now.

$$SCAL1(K, \alpha, q, e, v) := \left\| \begin{array}{l} \text{if } \alpha > 0 \\ \left\| E \leftarrow v - 2 \cdot \operatorname{atan} \left( \frac{e \cdot \sin(v)}{1 + \sqrt{1 - e^2} + e \cdot \cos(v)} \right) \right\| \\ s \leftarrow \frac{E}{\sqrt{\alpha}} \\ \text{else} \\ \left\| \left\| w \leftarrow \frac{1}{K} \cdot \sqrt{\frac{q}{1 + e}} \cdot \tan\left(\frac{v}{2}\right) \\ \text{if } \alpha = 0 \\ \left\| s \leftarrow 2 \cdot w \\ \text{else} \\ \left\| s \leftarrow 2 \cdot w \\ \text{else} \\ \right\| \\ s \leftarrow \frac{E}{\sqrt{-\alpha}} \\ \left\| s \leftarrow \frac{E}{\sqrt{-\alpha}} \\ \right\| \\ \left\| s \leftarrow \frac{E}{\sqrt{-\alpha}} \\ \right\| \\ s \leftarrow \frac{E}{\sqrt{-\alpha}} \\ \left\| s \leftarrow \frac{E}{\sqrt{-\alpha}} \\ \left\| s \leftarrow \frac{E}{\sqrt{-\alpha}} \\ \right\| \\ s \leftarrow \frac{E}{\sqrt{-\alpha}} \\ \left\| s \leftarrow \frac{E}{\sqrt{-\alpha}} \\ s \leftarrow \frac{E}{\sqrt{-\alpha}} \\ s \leftarrow \frac{E}{\sqrt{-\alpha}} \\ \left\| s \leftarrow \frac{E}{\sqrt{-\alpha}} \\ s \leftarrow \frac{E}{\sqrt{-\alpha}$$

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Finally, now, we define function **PVCO**.

(Note that in **PVCO**, as defined in this document, the subscripts of the **P**, **Q**, and **W** vectors range from 1 through 3 rather than from 0 through 2. Also, the subscripts of **c** range from 1 through 4 rather than from 0 through 3.)

$$PVCO(K, r, v) \coloneqq \begin{vmatrix} rmag \leftarrow \sqrt{r \cdot r} \\ h \leftarrow r \times v \\ hmag \leftarrow \sqrt{h \cdot h} \\ W \leftarrow \frac{h}{hmag} \\ E \leftarrow \frac{v \cdot v}{2} - \frac{K^2}{rmag} \\ a \leftarrow -2 \cdot E \\ p \leftarrow \frac{hmag^2}{K^2} \\ e \leftarrow \sqrt{1.0 - a \cdot p \cdot K^2} \\ q \leftarrow \frac{p}{1 + e} \\ U \leftarrow \frac{r}{rmag} \\ V \leftarrow W \times U \\ v \leftarrow \text{angle} \left(\frac{hmag}{K^2} \cdot v \\ P \leftarrow \cos(v) \cdot U - \sin(v) \cdot v + \cos(v) \\ i \leftarrow a \cos(W_3) \\ \Omega \leftarrow \text{angle} \left(-W_2, W_1 \\ \omega \leftarrow \text{angle} \left(Q_3, P_3\right) \\ s \leftarrow SCAL1(K, a, q, q) \\ c \leftarrow C(a \cdot s^2) \\ \Delta t \leftarrow q \cdot s + K^2 \cdot e \cdot s^2 \\ f = 1 \\ \end{vmatrix}$$

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$$\begin{aligned} h \leftarrow r \times v \\ hmag \leftarrow \sqrt{h \cdot h} \\ W \leftarrow \frac{h}{hmag} \\ E \leftarrow \frac{v \cdot v}{2} - \frac{K^2}{rmag} \\ a \leftarrow -2 \cdot E \\ p \leftarrow \frac{hmag^3}{k^2} \\ e \leftarrow \sqrt{1.0 - a \cdot p \cdot K^{-2}} \\ q \leftarrow \frac{p}{1 + e} \\ U \leftarrow \frac{r}{rmag} \\ V \leftarrow W \times U \\ v \leftarrow angle \left(\frac{hmag}{K^2} \cdot v \cdot V - 1.0, \frac{hmag}{K^2} \cdot v \cdot U\right) \\ P \leftarrow \cos(v) \cdot U - \sin(v) \cdot V \\ Q \leftarrow \sin(v) \cdot U + \cos(v) \cdot V \\ i \leftarrow a \cos(W_3) \\ \Omega \leftarrow angle \left(-W_2, W_1\right) \\ o \leftarrow angle \left(Q_2, P_3\right) \\ s \leftarrow SCALI(K, a, q, e, v) \\ c \leftarrow C(a \cdot s^2) \\ dt \leftarrow q \cdot s + K^2 \cdot e \cdot s^3 \cdot c_4 \\ \left[\begin{array}{c} q \\ e \\ i \cdot Deg PerRad \\ B \cdot Deg PerRad \\ B \cdot Deg PerRad \\ dt \end{array}\right] \end{aligned}$$

We now invoke **PVCO** and place its output into array **CONIC**.  $CONIC := PVCO(K, r_1, v_1)$   $CONIC = \begin{bmatrix} 0.49327855 \\ 4.44695566 \\ 35.93334847 \\ 6.46163765 \\ 125.76261396 \\ -187.4733701 \end{bmatrix}$ 

We should note that the position vector input to **PVCO** must have units of E.R. and the velocity vector must have units of E.R. per minute. We summarize the weighted batch least squares orbital solution as follows.

$(CONIC_1 - 1) \cdot a_e = -3231.93882$	Perigee height in km, relative to spherical Earth figure.
<i>CONIC</i> <sub>2</sub> = 4.44695566	Path eccentricity.
<i>CONIC</i> <sub>3</sub> = 35.93335	Path inclination, in degrees.
$CONIC_{4} = 6.46164$	Right ascension of ascending node, in degrees.
$CONIC_{5} = 125.76261$	Argument of perigee, in degrees.
$CONIC_{6} = -187.47337$	Time of flight from perigee to epoch, in minutes.
determination, it would be more accur	a spherical Earth figure, but for a closest approach ate to have the actual height of the object above its at the instant of perigee. We calculate this now.
$f \coloneqq \frac{1}{298.26}$	Earth's polar vs. equatorial flattening factor.
$e_e \coloneqq \sqrt{2 \cdot f - f^2}$	Eccentricity of Earth's meridional reference ellipse.

We define function **GRT**, which inputs space object's position vector and outputs the geodetic latitude of the subsatellite point (subpoint), and the object's height above the subpoint.

$$\begin{aligned} GRT(r) &\coloneqq \left\| rmag \leftarrow \sqrt{r \cdot r} \\ \delta \leftarrow \operatorname{asin} \left( \frac{r_{3}}{rmag} \right) \\ \phi_{c} \leftarrow \delta \\ \text{for } j \in 1 \dots 4 \\ \left\| r_{s} \leftarrow \frac{\sqrt{1 - e_{e}^{2}}}{\sqrt{1 - (e_{e} \cdot \cos(\phi_{c}))^{2}}} \\ \phi_{s} \leftarrow \operatorname{atan} \left( \frac{\tan(\phi_{c})}{1 - e_{e}^{2}} \right) \\ H_{s} \leftarrow \sqrt{rmag^{2} - (r_{s} \cdot \sin(\phi_{s} - \phi_{c}))^{2}} - r_{s} \cdot \cos(\phi_{s} - \phi_{c}) \\ \phi_{c} \leftarrow \delta - \operatorname{asin} \left( \frac{H_{s} \cdot \sin(\phi_{s} - \phi_{c})}{rmag} \right) \\ \left[ \phi_{s} \\ H_{s} \right] \\ \Delta t \coloneqq -CONIC_{6} \\ M \coloneqq FG\left( K, r_{1}, \frac{v_{l}}{K}, \Delta t \right) \\ f_{r} \coloneqq M_{1,4} \\ g_{r} \coloneqq M_{2,4} \\ r \coloneqq f_{r} \cdot r_{l} + g_{r} \cdot \frac{v_{l}}{K} \\ LatHt \coloneqq GRT(r) \end{aligned}$$

Geodetic latitude,  $\phi_{s}$ , and height above spheroid,  $H_{s}$ , at time of perigee passage:

$$LatHt_{,} \cdot DegPerRad = 28.76523$$
 (degrees)

$$LatHt \cdot a_e = -3227.04474$$
 (km)

7. Repeat Steps 1-6, by clicking on "Calculate Worksheet", until convergence is obtained.

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[10] Gaussian constant for Earth with units of E.R. <sup>3/2</sup> /min. Based here on updated WGS-84 value of GM = $3.986004415*10^8 \text{ m}^3$ / sec <sup>2</sup> . See David A. Vallado, <i>Fundamentals of Astrodynamics and Applications</i> , 5th Edition (2022), Microcosm Press, Torrance, CA USA, p. 132.

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